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Monotone Operators in Banach Space and Nonlinear Partial Differential Equations

R. E. Showalter



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ABSTRACT. This book is about monotone operators and nonlinear semigroup theory and their application to partial differential equations. The primary readership is advanced graduate students in mathematics or engineering science and researchers in partial differential equations or related analysis.

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Preface

The objectives in this monograph are to present some topics from the theory of monotone operators and nonlinear semigroup theory which are directly applicable to the existence and uniqueness theory of initial-boundary-value problems for partial differential equations and to construct such operators as realizations of those problems in appropriate function spaces. A highlight of the presentation will be the large number and variety of examples which are introduced to illustrate this connection between the theory of nonlinear operators and partial differential equations. These include primarily semilinear or quasilinear equations of elliptic or of parabolic type, degenerate cases with change of type, related systems and variational inequalities, and spatial boundary conditions of the usual Dirichlet, Neumann, Robin or of dynamic type. The discussions of evolution equations include the usual initial-value problems as well as periodic or more general nonlocal constraints, history-value problems, those which may change type due to a possibly vanishing coefficient of the time derivative, and other implicit evolution equations or systems including hysteresis models. The scalar conservation law and semilinear wave equations are briefly mentioned, and hyperbolic systems arising from vibrations of elastic-plastic rods are developed. The origins of a representative sample of such problems is given in the Appendix.

This is the place to *begin* study of a particular problem. Once a proper setting has been established for a given problem to be well-posed, one can then proceed to investigate those properties of solutions which distinguish the problem, such as regularity, asymptotic behavior, numerical analysis, stability, special properties, controllability,.... None of these topics will be discussed here. The objective is rather to develop for the reader an instinct for the *right* place (or places) to look for a solution and the *right* techniques to use to establish existence-uniqueness in appropriate function spaces for a broad class of problems. Much attention has been devoted to develop the connection between the abstract theory and the specific problems for partial differential equations.

The work is arranged in four chapters and an appendix, and each of these is divided into numbered sections. All results are formally stated as Theorems, Propositions, Lemmas, or Corollaries which are independently numbered in the respective category by their section number and order within that section. Thus, in Section 4 of Chapter I the second Proposition is called Proposition 4.2, and it is referenced that way within Chapter I. From any other chapter it will be recalled as Proposition I.4.2. Most examples are named alphabetically within their section, so the third example of Section I.4 is Example 4.C, and from outside Chapter I it is called Example I.4.C.

The beginning chapter gives a casual but technically precise overview of the subject in the case of linear problems in one spatial dimension. Most of the major notions appear here in this simple setting, including the Lax-Milgram-Lions Theorem and the Hille-Yosida Theorem, and they are motivated by the classical Dirichlet and Neumann boundary-value problems and by various initial-boundary-value problems, respectively. The second and third chapters begin with the theory of monotone nonlinear operators from a reflexive Banach space to its dual and the solution of corresponding stationary or time-dependent problems with such operators. The remainder of each consists of the development of the applications to appropriate classes of problems. The fourth chapter is concerned with the theory of accretive operators in a single Hilbert or Banach space. Some very useful topics of convex analysis are developed independently in both the second and fourth chapters. Each chapter contains a wealth of examples of initial-boundary-value problems for partial differential equations to which the abstract results apply. The Appendix contains descriptions of the derivation of the partial differential equations and initial-boundary-value problems for heat transport, for flow through porous media, and for simple vibration problems of mechanics. These model problems begin from the most elementary considerations of the physics and proceed to many types of boundary conditions, the Stefan free-boundary problem, the porous medium equation, systems to describe diffusion in composite (fissured) media, and models of plasticity and hysteresis. These are intended to illustrate the variety of nonlinearities that are covered by the monotone theory.

There is much independence and intentional repetition between the four chapters, so one can frequently find a short path to a later section. Chapter I is not logically necessary for anything that follows, but every reader should at least look through it. For the less mathematically prepared it provides a quick but self-contained introduction to some linear functional analysis topics, and to the more seasoned reader it can serve as an orientation to the developments to follow. For one who will be satisfied with problems in one spatial dimension, such as two-point boundary-value problems and parabolic or wave equations in the plane, it contains the construction of the elementary function spaces necessary to bypass the technical Sections II.3 and II.4. A rather complete exposition of the linear version of this material is contained in Chapter I and the first three sections of Chapter III. Chapter III depends heavily on Chapter II. Chapter IV is essentially independent of all other chapters. It needs only the motivation from Section I.4, and Section II.9 is used in Section IV.9.

This work was written primarily for advanced graduate students in mathematics or engineering science, researchers in partial differential equations and related numerical analysis, applied mathematics, control theory or dynamical systems for whom a precise existence-uniqueness theory for initial-boundary-value problems is of interest. The completeness and level of the presentation should make the book much easier to read than most research papers on the subject. The primary prerequisite for the reader is a previous acquaintance with L^p spaces and the notions of *continuous* and *linear*. The author has taught variations of this material to such audiences for the past 20 years, with classroom presentation and style modified accordingly for the specific group. This is the material that has proven to be most universally applicable and of interest to the students and seminar participants, many of whom had research interests outside of mathematics. It is not written as a traditional text, but rather as a guidebook, to lead the reader along the more

accessible valley trails, to provide a view of the steep terrain of the higher peaks and an appreciation of those who travelled them before us.

No attempt has been made to include references to all the research papers in which this material was originally developed or to recount the history of these developments. The Bibliography contains references only to books which contain portions of this material, and one can consult these for further discussion of any one topic or for references to the various sources and versions of the history. If the reader develops here a sufficient appreciation of the mathematical structure and power of these methods to pursue one of these references, then this work will have served its purpose.

It is a pleasure to acknowledge a debt of personal gratitude to the people who contributed to the formulation and preparation of this book at its various stages. The text benefited substantially from the constructive comments of various students who read parts of the manuscript at a formative stage for both, and I take great pride in the success that many of them have experienced in their research. Betty Banner was responsible for the arduous metamorphosis of my scribbled notes into an early form of the manuscript. Margaret Combs graciously shared her immense T_EXpertise and gave invaluable professional assistance at every stage of the preparation. Her enthusiasm in the final stages of the project contributed to its timely completion. Małgorzata Peszyńska carefully and patiently read the final version of the manuscript. The reader will benefit from her many helpful remarks and suggestions for improvement of the exposition. For myself, I claim sole responsibility for any remaining errors or offending omissions.

Finally, the author is grateful to the very professional and seemingly tireless staff of the National Science Foundation for the effective and productive support of the research reported here.

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PDE Examples by Type

First Order Equations

initial value problem	I.5.A, II.9.A, III.2.A, IV.2.D, IV.3, IV.9.A(SCL)
boundary value problem	I.4.B, II.9.A
systems, inequalities	A.3

Elliptic Equations

linear	I.3.A,B,E, I.4.C,D, II.9
semilinear	II.8.D,E,F, II.9.B, IV.2.E,F
quasilinear	II.5.A,B,C,D,E, II.6
inequalities	I.3.C,D, II.6.A,B,C,D
systems	II.5.F,G,H,I

Parabolic Equations

linear	I.5.B, II.2.B, A.1, A.2
semilinear	III.6.C (PME), IV.2.G, IV.6.B (PME), IV.7, IV.9.B,C,D (PME), A.1, A.2
quasilinear	III.4.A,B, III.6.A, IV.5.A,B, A.2
inequalities	III.7.A,B,C,D, IV.5.B, IV.9.C, A.1
degenerate	III.3.A,B, III.6.B, IV.6.A (FME), IV.6, A.1
systems	III.5, III.7.D, IV.9.C,D, A.1, A.2, A.3

Wave Equations

linear	I.6
semilinear	IV.6, A.3
systems	A.3

CHAPTER I

Linear Problems ... an Introduction

I.1. Boundary-Value Problems in 1-D

We begin by considering two classical boundary value problems. Let $H \equiv L^2(a, b)$, the space of (equivalence classes of) square-summable functions on the real interval (a, b) . By $u' \in H$ we mean that u is an anti-derivative of a function in $L^2(a, b)$, hence, it is absolutely continuous and the classical derivative $u'(x)$ exists at a.e. point x in (a, b) . We shall consider the two following boundary-value problems. Let $c \in \mathbb{R}$ and $F \in H$ be given. The *Dirichlet problem* is to find

$$u \in H : -u'' + cu = F \text{ in } H, \quad u(a) = u(b) = 0 ,$$

and the *Neumann problem* is to find

$$u \in H : -u'' + cu = F \text{ in } H, \quad u'(a) = u'(b) = 0 .$$

An implicit requirement of each of these classical formulations is that $u'' \in H$. This smoothness condition can be relaxed: multiply the equation by $v \in H$ and integrate. If also $v' \in H$ we obtain the following by an integration-by-parts. Let $V_0 \equiv \{v \in H : v' \in H \text{ and } v(a) = v(b) = 0\}$; a solution of the Dirichlet problem is characterized by

$$u \in V_0 \text{ and } \int_a^b (u'v' + cuv) dx = \int_a^b Fv dx , \quad v \in V_0 .$$

Similarly, a solution of the Neumann problem satisfies

$$u \in V_1 \text{ and } \int_a^b (u'v' + cuv) dx = \int_a^b Fv dx , \quad v \in V_1 ,$$

where $V_1 \equiv \{v \in H : v' \in H\}$. These are the corresponding *weak formulations* of the respective problems. We shall see directly that they are actually equivalent to their respective classical formulations. Moreover we see already the primary ingredients of the variational theory:

- (a) *Functionals*. Each function, e.g., $F \in H$, is identified with a functional, $\tilde{F} : H \rightarrow \mathbb{R}$, defined by $\tilde{F}(v) \equiv \int_a^b Fv dx$, $v \in H$. This identification is achieved by way of the L^2 scalar product. For a pair $u \in V_1$, $v \in V_0$ an integration by parts shows $\tilde{u}'(v) = -\tilde{u}(v')$. Thus, for this identification of functions with functionals to be consistent with the usual differentiation of functions, it is *necessary* to define the *generalized derivative* of a functional f by $\partial f(v) \equiv -f(v')$, $v \in V_0$.

- (b) *Function Spaces.* From $L^2(a, b)$ and the generalized derivative ∂ we construct the Sobolev space $H^1(a, b) \equiv \{v \in L^2(a, b) : \partial v \in L^2(a, b)\}$. We shall see directly that $H^1(a, b)$ is a Hilbert space with the scalar product

$$(u, v)_{H^1} \equiv \int_a^b (\partial u \partial v + uv) dx$$

and corresponding norm $\|u\|_{H^1} = (u, u)_{H^1}^{1/2}$, and each of its members is absolutely continuous, hence, $u(x) - u(y) = \int_y^x \partial u$ for $u \in H^1(a, b)$, $a < y < x < b$. This space arises naturally above in the Neumann problem; we denote by $H_0^1(a, b)$ the subspace $\{v \in H^1(a, b) : v(a) = v(b) = 0\}$ which occurs in the Dirichlet problem.

- (c) *Forms.* Each of our weak formulations is phrased as

$$u \in V : a(u, v) = f(v), \quad v \in V,$$

where V is the appropriate Hilbert space (either H_0^1 or H^1), $f = \tilde{F}$ is a continuous linear functional on V , and $a(\cdot, \cdot)$ is the bilinear form on V defined by

$$a(u, v) = \int_a^b (\partial u \partial v + cuv) dx, \quad u, v \in V.$$

This form is bounded or *continuous* on V : there is a $C > 0$ such that

$$(1.1) \quad |a(u, v)| \leq C \|u\|_V \|v\|_V, \quad u, v \in V.$$

Moreover, it is *V-coercive*, i.e., there is a $c_0 > 0$ for which

$$(1.2) \quad |a(v, v)| \geq c_0 \|v\|_V^2, \quad v \in V$$

in the case of $V = H^1(a, b)$ if (and only if!) $c > 0$ and in the case of $V = H_0^1(a, b)$ for any $c > -2/(b - a)^2$. (This last inequality follows from (1.3) below but it is not the optimal constant.) We shall see that the weak formulation constitutes a well-posed problem whenever the bilinear form is bounded and coercive.

First we focus on the notion of a generalized derivative of functions and, even more generally, of functionals. These notions are fundamental to the construction of Sobolev spaces. A non-standard aspect of our presentation is that we shall refer to any linear functional (not necessarily continuous) on test functions as a *distribution*. Since all analysis is done in Hilbert subspaces of such functionals, no topological notions are needed for the whole space of functionals.

Let $-\infty \leq a < b \leq +\infty$. The *support* of a function $\varphi : (a, b) \rightarrow \mathbb{R}$ is the closure in (a, b) of the set $\{x \in (a, b) : \varphi(x) \neq 0\}$. Then $C_0^\infty(a, b)$ is the linear space of those infinitely differentiable functions $\varphi : (a, b) \rightarrow \mathbb{R}$ each of which has compact support in (a, b) . An example is given by

$$\varphi(x) = \begin{cases} \exp[-1/(1 - |x|^2)] & |x| < 1 \\ 0 & |x| \geq 1. \end{cases}$$

A linear functional, $T : C_0^\infty(a, b) \rightarrow \mathbb{R}$, is called a *distribution* on (a, b) ; the linear space of all distributions is the algebraic dual $C_0^\infty(a, b)^*$ of $C_0^\infty(a, b)$. We shall refer to $C_0^\infty(a, b)$ as the space of *test functions* on (a, b) .

A measurable function $u : (a, b) \rightarrow \mathbb{R}$ is *locally integrable* on (a, b) if for every compact set $K \subset (a, b)$, we have $\int_K |u| dx < \infty$. The space of all such (equivalence

classes of) functions is denoted by $L^1_{\text{loc}}(a, b)$. Suppose u is (a representative of) an element of $L^1_{\text{loc}}(a, b)$. Then we define a corresponding distribution $\tilde{u}$ by

$$\tilde{u}(\varphi) = \int_a^b u(x)\varphi(x) dx, \quad \varphi \in C_0^\infty(a, b).$$

Note that $\tilde{u}$ is independent of the representative and that the function $u \mapsto \tilde{u}$ is linear from L^1_{loc} to $C_0^{\infty*}$.

LEMMA 1.1. *If $\tilde{u} = 0$ then $u = 0$.*

SKETCH OF PROOF. We have $\int u\varphi = 0$ for all $\varphi \in C_0^\infty$. Extend this to hold for all continuous functions with compact support by a convolution approximation. (See II.3.) Then extend to all bounded measurable functions with compact support by Lebesgue theory. Finally, choose $\varphi_\lambda(x) = u(x)$ for $|x| \leq \lambda$ and $|u(x)| \leq \lambda$, set $\varphi_\lambda(x) = \lambda$ for $u_\lambda(x) \geq \lambda$, $\varphi_\lambda(x) = -\lambda$ for $u(x) \leq -\lambda$, and $\varphi_\lambda(x) = 0$ for $|x| \geq \lambda$. Then let $\lambda \rightarrow \infty$. $\square$

PROPOSITION 1.1. *The mapping $u \mapsto \tilde{u}$ of $L^1_{\text{loc}}(a, b)$ into $C_0^\infty(a, b)^*$ is linear and one-to-one.*

PROOF. The kernel of this map is the zero element. $\square$

We call $\{\tilde{u} : u \in L^1_{\text{loc}}(G)\}$ the *regular distributions*. Two examples in $C_0^\infty(\mathbb{R})^*$ are the *Heaviside functional*

$$\tilde{H}(\varphi) = \int_0^\infty \varphi, \quad \varphi \in C_0^\infty(\mathbb{R}),$$

obtained from the *Heaviside function*: $H(x) = 1$ if $x > 0$ and $H(x) = 0$ for $x < 0$, and the constant functional

$$T(\varphi) = \int_{\mathbb{R}} \varphi, \quad \varphi \in C_0^\infty(\mathbb{R})$$

given by $T = \tilde{1}$. An example of a non-regular distribution is the *Dirac functional* given by

$$\delta(\varphi) = \varphi(0), \quad \varphi \in C_0^\infty(\mathbb{R}).$$

According to Proposition 1.1 the space of distributions is so large that it contains all functions with which we shall be concerned, i.e., it contains L^1_{loc} . Such a large space was constructed by taking the dual of the “small” space C_0^∞ . Next we shall take advantage of the linear differentiation operator on C_0^∞ to construct a corresponding generalized differentiation operator on the dual space of distributions. Moreover, we shall define the derivative of a distribution in such a way that it is consistent with the classical derivative on functions. Let D denote the classical derivative, $D\varphi = \varphi'$, when it is defined at a.e. point of the domain of φ . As we observed above, if we want to define a generalized derivative ∂T of a distribution T so that for each $u \in C^\infty(a, b)$ we have $\partial \tilde{u} = \widetilde{(Du)}$, that is,

$$\partial \tilde{u}(\varphi) = - \int_a^b u \cdot D\varphi = -\tilde{u}(D\varphi), \quad \varphi \in C_0^\infty(a, b),$$

then we must define ∂ as follows.

DEFINITION. For each distribution $T \in C_0^\infty(a, b)^*$ the derivative $\partial T \in C_0^\infty(a, b)^*$ is defined by

$$\partial T(\varphi) = -T(D\varphi) , \quad \varphi \in C_0^\infty(a, b) .$$

Note that $D : C_0^\infty(a, b) \rightarrow C_0^\infty(a, b)$ and $T : C_0^\infty(a, b) \rightarrow \mathbb{R}$ are both linear, so $\partial T : C_0^\infty(a, b) \rightarrow \mathbb{R}$ is clearly linear. Also ∂ is just the negative of the dual D^* of the linear map D , $\partial = -D^* : C_0^\infty(a, b)^* \rightarrow C_0^\infty(a, b)^*$. Since ∂ is defined on all distributions, it follows that every distribution has derivatives of all orders. Specifically, every $u \in L_{\text{loc}}^1$ has derivatives in $C_0^\infty(a, b)^*$ of all orders.

EXAMPLE 1.A. Let f be continuously differentiable on $\mathbb{R}$. Then we have

$$\partial \tilde{f}(\varphi) = -\tilde{f}(D\varphi) = -\int f D\varphi dx = \int Df \varphi = \widetilde{(Df)}(\varphi)$$

for $\varphi \in C_0^\infty(\mathbb{R})$. The third equality follows from integration-by-parts and all other equalities are definitions. Thus the generalized derivative coincides with the classical derivative on smooth functions. Of course the definition was rigged to make this occur.

EXAMPLE 1.B. Let $r(x) = xH(x)$ where $H(x)$ is given above. For this piecewise-differentiable function we have

$$\partial \tilde{r}(\varphi) = -\int_0^\infty x D\varphi(x) dx = \int_0^\infty \varphi(x) dx = \tilde{H}(\varphi) , \quad \varphi \in C_0^\infty(\mathbb{R}) ,$$

so $\partial \tilde{r} = \tilde{H}$ even though $Dr(0)$ does not exist.

EXAMPLE 1.C. For the piecewise-continuous function H we have

$$\partial \tilde{H}(\varphi) = -\int_0^\infty D\varphi(x) dx = \varphi(0) = \delta(\varphi) , \quad \varphi \in C_0^\infty(\mathbb{R}) ,$$

so $\partial \tilde{H} = \delta$, the non-regular Dirac functional. More generally, let $f : \mathbb{R} \rightarrow \mathbb{R}$ be absolutely continuous in a neighborhood of each $x \neq 0$ and have one-sided limits $f(0^+)$ and $f(0^-)$ from the right and left, respectively, at 0. Then we obtain

$$\begin{aligned} \partial \tilde{f}(\varphi) &= -\int_0^\infty f D\varphi - \int_{-\infty}^0 f D\varphi = \int_0^\infty (Df)\varphi + f(0^+)\varphi(0) \\ &\quad + \int_{-\infty}^0 (Df)\varphi - f(0^-)\varphi(0) = \widetilde{(Df)}(\varphi) + \sigma_0(f)\varphi(0) , \quad \varphi \in C_0^\infty , \end{aligned}$$

where $\sigma_0(f) = f(0^+) - f(0^-)$ is the jump in f at 0. That is, $\partial \tilde{f} = \widetilde{Df} + \sigma_0(f)\delta$, and this formula can be repeated if Df satisfies the preceding conditions on f :

$$\partial^2 \tilde{f} = \widetilde{(D^2 f)} + \sigma_0(Df)\delta + \sigma_0(f)\partial\delta .$$

For example we have

$$\partial(H \cdot \sin) = H \cdot \cos , \quad \partial(H \cdot \cos) = -H \cdot \sin + \delta .$$

Before discussing further the interplay between ∂ and D we note that a distribution T on $\mathbb{R}$ is constant if and only if $T = \tilde{c}$ for some $c \in \mathbb{R}$, i.e.,

$$T(\varphi) = c \int_{\mathbb{R}} \varphi, \quad \varphi \in C_0^\infty.$$

This occurs exactly when T depends only on the mean value of each φ . This observation is the key to the description of primitives or anti-derivatives of a given distribution. Suppose we are given a distribution S on $\mathbb{R}$; does there exist a primitive, a distribution T such that $\partial T = S$? That is, do we have a distribution T for which

$$T(D\psi) = -S(\psi), \quad \psi \in C_0^\infty(\mathbb{R})?$$

LEMMA 1.2.

- (a) $\{D\psi : \psi \in C_0^\infty(\mathbb{R})\} = \{\zeta \in C_0^\infty(\mathbb{R}) : \int \zeta = 0\}$ and the correspondence is given by $\psi(x) = \int_{-\infty}^x \zeta$.
- (b) Denote the space in (a) by $\mathcal{H}$ and let $\varphi_0 \in C_0^\infty(\mathbb{R})$ with $\int \varphi_0 = 1$. Then each $\varphi \in C_0^\infty(\mathbb{R})$ can be uniquely written as $\varphi = \zeta + c\varphi_0$ with $\zeta \in \mathcal{H}$, and this occurs when $c = \int \varphi$.

PROPOSITION 1.2.

- (a) For each distribution S there is a distribution T with $\partial T = S$.
- (b) If T_1, T_2 are distributions with $\partial T_1 = \partial T_2$, then $T_1 = T_2 + \text{constant}$.

PROOF.

- (a) Define T on $\mathcal{H}$ by $T(\zeta) = -S(\psi)$, $\zeta \in \mathcal{H}$, $\psi(x) = \int_{-\infty}^x \zeta$, and extend to all of $C_0^\infty(\mathbb{R})$ by $T(\varphi_0) = 0$.
- (b) If $\partial T = 0$, then $T(\varphi) = T(\zeta + c\varphi_0) = T(\varphi_0) \int \varphi$, so $T = T(\varphi_0)\tilde{1}$ is a constant. $\square$

COROLLARY 1.1. If T is a distribution on $\mathbb{R}$ with $\partial T \in L_{\text{loc}}^1(\mathbb{R})$, then $T = \tilde{f}$ for some absolutely continuous f , and $\partial T = \widetilde{Df}$.

PROOF. Note first that if f is absolutely continuous then $Df(x)$ is defined for a.e. $x \in \mathbb{R}$ and $Df \in L_{\text{loc}}^1(\mathbb{R})$ with $\widetilde{Df} = \partial \tilde{f}$ as before. In the converse situation of Corollary 1.1, let $g \in L_{\text{loc}}^1$ with $\tilde{g} = \partial T$ and define $h(x) = \int_0^x g$, $x \in \mathbb{R}$. Then h is absolutely continuous, $\partial(T - \tilde{h}) = 0$, so Proposition 1.2 shows $T = \tilde{h} + \tilde{c}$ for some $c \in \mathbb{R}$. Thus $T = \tilde{f}$ with $f(x) = h(x) + c$, $x \in \mathbb{R}$. $\square$

COROLLARY 1.2. The weak formulations of the Dirichlet and Neumann problems are equivalent to the original formulations.

Finally we describe the spaces that naturally arise in the consideration of such boundary-value problems. The Sobolev space $H^1(a, b)$ is given by

$$H^1(a, b) = \{u \in L^2(a, b) : \partial u \in L^2(a, b)\}$$

where we have identified $u \cong \tilde{u}$. Thus each $u \in H^1(a, b)$ is absolutely continuous with

$$u(x) - u(y) = \int_y^x \partial u, \quad a \leq x, y \leq b.$$

This gives the Hölder continuity estimate (see (2.1) below)

$$|u(x) - u(y)| \leq |x - y|^{1/2} \|\partial u\|_{L^2(a,b)}, \quad u \in H^1(a,b), a \leq x, y \leq b.$$

If also we have $u(a) = 0$ then there follow

$$(1.3) \quad |u(x)| \leq (b-a)^{1/2} \|\partial u\|_{L^2(a,b)}, \quad a \leq x \leq b,$$

$$(1.4) \quad \|u\|_{L^2(a,b)} \leq ((b-a)/\sqrt{2}) \|\partial u\|_{L^2(a,b)},$$

and such estimates also hold for those $u \in H^1(a,b)$ with $u(b) = 0$. Let $\lambda(x) = (x-a)(b-a)^{-1}$ and $u \in H^1(a,b)$. Then $\lambda u \in H^1(a,b)$ and $\partial(\lambda u) = \lambda \partial u + (b-a)^{-1}u$, so $\|\partial(\lambda u)\|_{L^2} \leq \|\partial u\|_{L^2} + (b-a)^{-1}\|u\|_{L^2}$. The same holds for $\partial((1-\lambda)u)$ so by writing $u = \lambda u + (1-\lambda)u$ we obtain

$$(1.5) \quad \max\{|u(x)| : a \leq x \leq b\} \leq 2(b-a)^{1/2} \|\partial u\|_{L^2} + 2(b-a)^{-1/2} \|u\|_{L^2}, \quad u \in H^1(a,b).$$

This simple estimate will be very useful throughout this introductory chapter.

More generally, we define for each integer $k \geq 1$ the *Sobolev space*

$$H^k(a,b) = \{u \in L^2(a,b) : \partial^j u \in L^2(a,b)\}, \quad 1 \leq j \leq k.$$

Estimates analogous to those above can be easily obtained in appropriate subspaces.

I.2. Variational Method in Hilbert Space

Our objective is to review certain topics in the elementary theory of Hilbert space which lead directly to abstract variational or weak formulations of boundary value problems. Let V be a linear space over the reals $\mathbb{R}$ and the function $x, y \mapsto (x, y)$ from $V \times V$ to $\mathbb{R}$ be a *scalar product*. That is, $(x, x) > 0$ for non-zero $x \in V$, $(x, y) = (y, x)$ for $x, y \in V$, and for each $y \in V$ the function $x \mapsto (x, y)$ is linear from V to $\mathbb{R}$. For each pair $x, y \in V$ it follows that

$$(2.1) \quad |(x, y)|^2 \leq (x, x)(y, y).$$

To see this, we note that

$$0 \leq (tx + y, tx + y) = t^2(x, x) + 2t(x, y) + (y, y), \quad t \in \mathbb{R},$$

and so the discriminant of the quadratic must be non-positive. From (2.1) it follows that $\|x\| \equiv (x, x)^{1/2}$, $x \in V$, defines a *norm* on V : $\|x\| \geq 0$, $\|tx\| = |t| \|x\|$, and $\|x + y\| \leq \|x\| + \|y\|$ for $x, y \in V$ and $t \in \mathbb{R}$. Thus every scalar product induces a norm and corresponding *metric* $d(x, y) = \|x - y\|$. A sequence $\{x_n\}$ *converges* to x in V if $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$. This is denoted by $\lim_{n \rightarrow \infty} x_n = x$. A convergent sequence is always *Cauchy*: $\lim_{m, n \rightarrow \infty} \|x_m - x_n\| = 0$. The space V with norm $\|\cdot\|$ is *complete* if each Cauchy sequence is convergent in V . A complete normed linear space is a *Banach space*, and a complete scalar product space is a *Hilbert space*.

Some familiar examples of Hilbert spaces include Euclidean space $\mathbb{R}^m = \{\vec{x} = (x_1, x_2, \dots, x_m) : x_j \in \mathbb{R}\}$ with $(\vec{x}, \vec{y}) = \sum_{j=1}^m x_j y_j$, the sequence space $\ell^2 = \{\vec{x} = \{x_1, x_2, x_3, \dots\} : \sum_{j=1}^{\infty} |x_j|^2 < \infty\}$ with $(\vec{x}, \vec{y}) = \sum_{j=1}^{\infty} x_j y_j$, and the Lebesgue space $L^2(\Omega) = \{\text{equivalence classes of measurable functions } f : \Omega \rightarrow \mathbb{R} : \int_{\Omega} |f|^2 d\mu$

$< \infty\}$ with $(f, g) = \int_{\Omega} f(w)g(w) d\mu$, where (Ω, μ) is a measure space. Another example is the Sobolev space $H^1(a, b)$ with the scalar product

$$(u, v)_{H^1} = (u, v)_{L^2} + (\partial u, \partial v)_{L^2}, \quad u, v \in H^1(a, b).$$

To verify that this space is complete, let $\{u_n\}$ be a Cauchy sequence, so that both $\{u_n\}$ and $\{\partial u_n\}$ are Cauchy sequences in $L^2(a, b)$. Since $L^2(a, b)$ is complete there are $u, v \in L^2(a, b)$ for which $\lim u_n = u$ and $\lim \partial u_n = v$ in $L^2(a, b)$. For each $\varphi \in C_0^\infty(a, b)$ we have

$$-\int_a^b u_n \cdot D\varphi = \int_a^b \partial u_n \varphi, \quad n \geq 1,$$

so letting $n \rightarrow \infty$ shows $v = \partial u$. Thus $u \in H^1(a, b)$ and $\lim u_n = u$ in $H^1(a, b)$.

Let V_1 and V_2 be normed linear spaces with corresponding norms $\|\cdot\|_1, \|\cdot\|_2$. A function $T : V_1 \rightarrow V_2$ is *continuous at* $x \in V_1$ if $\{T(x_n)\}$ converges to $T(x)$ in V_2 whenever $\{x_n\}$ converges to x in V_1 . It is *continuous* if it is continuous at every x . For example, the norm is continuous from V_1 into $\mathbb{R}$. If T is linear, we shall also denote its value at x by Tx instead of $T(x)$.

PROPOSITION 2.1. *If $T : V_1 \rightarrow V_2$ is linear, the following are equivalent:*

- (a) *T is continuous at 0,*
- (b) *T is continuous at every $x \in V_1$,*
- (c) *there is a constant $K \geq 0$ such that $\|Tx\|_2 \leq K\|x\|_1$ for all $x \in V_1$.*

PROOF. Clearly (c) implies (b) by linearity and (b) implies (a). If (c) were false there would be a sequence $\{x_n\}$ in V_1 with $\|Tx_n\|_2 > n\|x_n\|_1$, but then $y_n \equiv \|Tx_n\|_2^{-1} x_n$ is a sequence which contradicts (a). $\square$

We shall denote by $\mathcal{L}(V_1, V_2)$ the set of all continuous linear functions from V_1 to V_2 ; these are called the *bounded linear functions* because of (c) above. Additional structure on this set is given as follows.

PROPOSITION 2.2. *For each $T \in \mathcal{L}(V_1, V_2)$ we have*

$$\begin{aligned} \|T\| &\equiv \sup\{\|Tx\|_2 : x \in V_1, \|x\|_1 \leq 1\} = \sup\{\|Tx\|_2 : \|x\|_1 = 1\} \\ &= \inf\{K > 0 : \|Tx\|_2 \leq K\|x\|_1, \quad x \in V_1\}, \end{aligned}$$

and this gives a norm on $\mathcal{L}(V_1, V_2)$. If V_2 is complete, then $\mathcal{L}(V_1, V_2)$ is complete.

PROOF. Consider the two numbers

$$\lambda = \sup\{\|Tx\|_2 : \|x\|_1 \leq 1\}, \quad \mu = \inf\{K > 0 : \|Tx\|_2 \leq K\|x\|_1, \quad x \in V_1\}.$$

If K is in the set defining μ , then for each $x \in V_1$ with $\|x\|_1 \leq 1$ we have $\|Tx\|_2 \leq K$, so $\lambda \leq K$. This holds for all such K so $\lambda \leq \mu$. If $x \in V_1$ with $\|x\|_1 > 0$ then $x/\|x\|_1$ is a unit vector and so $\|T(x/\|x\|_1)\|_2 \leq \lambda$. Thus $\|Tx\|_2 \leq \lambda\|x\|_1$ for all $x \neq 0$, and it clearly holds if $x = 0$, so we have $\mu \leq \lambda$. This establishes the equality of the three expressions for $\|T\|$, and it is easy to check that this defines a norm on $\mathcal{L}(V_1, V_2)$.

Suppose V_2 is complete and let $\{T_n\}$ be a Cauchy sequence in $\mathcal{L}(V_1, V_2)$. For each $x \in V_1$,

$$\|T_m x - T_n x\|_2 \leq \|T_m - T_n\| \|x\|_1$$

so $\{T_n x\}$ is Cauchy in V_2 , hence, convergent to a unique Tx in V_2 . This defines $T : V_1 \rightarrow V_2$ and it follows by continuity of addition and scalar multiplication that T is linear. Also

$$\|T_n x\|_2 \leq \|T_n\| \|x\|_1 \leq \sup\{\|T_m\|\} \|x\|_1$$

so letting $n \rightarrow \infty$ shows T is continuous with $\|T\| \leq \sup\{\|T_m\|\}$. Finally, to show $\lim T_n = T$, let $\varepsilon > 0$ and choose N so large that $\|T_m - T_n\| < \varepsilon$ for $m, n \geq N$. Then for each $x \in V_1$ $\|T_m x - T_n x\|_2 \leq \varepsilon \|x\|_1$, and letting $m \rightarrow \infty$ gives

$$\|Tx - T_n x\|_2 \leq \varepsilon \|x\|_1, \quad x \in V_1.$$

Thus, $\|T - T_n\| \leq \varepsilon$ for $n \geq N$. $\square$

As a consequence it follows that the *dual* $V' \equiv \mathcal{L}(V, \mathbb{R})$ of any normed linear space V is complete with the dual norm

$$\|f\|_{V'} \equiv \sup\{|f(x)| : x \in V, \|x\|_V \leq 1\}$$

for $f \in V'$.

Hereafter we let V denote a Hilbert space with norm $\|\cdot\|$, scalar product $(\cdot, \cdot)$, and dual space V' . A subset K of V is called *closed* if each $x_n \in K$ and $\lim x_n = x$ imply $x \in K$. The subset K is *convex* if $x, y \in K$ and $0 \leq t \leq 1$ imply $tx + (1-t)y \in K$. The following *minimization principle* is fundamental.

THEOREM 2.1. *Let K be a closed, convex, non-empty subset of the Hilbert space V , and let $f \in V'$. Define $\varphi(x) \equiv (1/2)\|x\|^2 - f(x)$, $x \in V$. Then there exists a unique*

$$(2.2) \quad x \in K : \varphi(x) \leq \varphi(y), \quad y \in K.$$

PROOF. Set $d \equiv \inf\{\varphi(y) : y \in K\}$ and choose $x_n \in K$ such that $\lim_{n \rightarrow \infty} \varphi(x_n) = d$. Then we obtain successively

$$\begin{aligned} d &\leq \varphi(1/2(x_m + x_n)) = (1/2)(\varphi(x_m) + \varphi(x_n)) - (1/8)\|x_n - x_m\|^2, \\ (1/4)\|x_n - x_m\|^2 &\leq \varphi(x_m) + \varphi(x_n) - 2d, \end{aligned}$$

and this last expression converges to zero. Thus $\{x_n\}$ is Cauchy, it converges to some $x \in V$ by completeness, and $x \in K$ since it is closed. Since φ is continuous, $\varphi(x) = d$ and x is a solution of (2.2). If x_1 and x_2 are both solutions of (2.2), the last inequality shows $(1/4)\|x_1 - x_2\| \leq d + d - 2d = 0$, so $x_1 = x_2$. $\square$

The solution of the minimization problem (2.2) can be characterized by a *variational inequality*. For $x, y \in V$ and $t > 0$ we have $(1/t)(\varphi(x + t(y-x)) - \varphi(x)) = (x, y-x) - f(y-x) + (1/2)t\|y-x\|^2$, so the *derivative* of φ at x in the direction $y-x$ is given by

$$\begin{aligned} \varphi'(x)(y-x) &= \lim_{t \rightarrow 0} (1/t)(\varphi(x + t(y-x)) - \varphi(x)) \\ (2.3) \quad &= (x, y-x) - f(y-x). \end{aligned}$$

An easy calculation shows the above equals $\varphi(y) - \varphi(x) + (x, y) - (1/2)\|x\|^2 - (1/2)\|y\|^2$, so (2.1) gives

$$(2.4) \quad \varphi'(x)(y-x) \leq \varphi(y) - \varphi(x), \quad x, y \in V.$$

Suppose x is a solution of (2.2). Since for each $y \in K$ we have $x + t(y - x) \in K$ for small $t > 0$, it follows from (2.3) that

$$x \in K : \varphi'(x)(y - x) \geq 0, \quad y \in K.$$

Conversely, for any such x it follows from (2.4) that it satisfies (2.2). Thus, we have shown that (2.2) is equivalent to

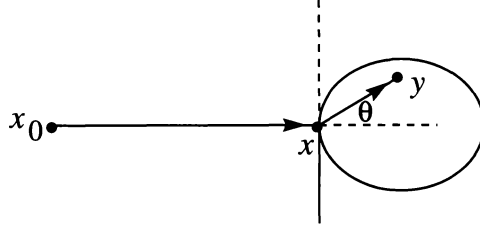
$$(2.5) \quad x \in K : (x, y - x) \geq f(y - x), \quad y \in K.$$

The equivalence of (2.2) and (2.5) is merely the fact that the point where a quadratic function takes its minimum is characterized by having a non-negative derivative in each direction into the set.

As an example, let $x_0 \in V$ and define $f \in V'$ by $f(y) = (x_0, y)$ for $y \in V$. Then $\varphi(x) = (1/2)(\|x - x_0\|^2 - \|x_0\|^2)$ so (2.2) means that x is that point of K which is closest to x_0 . Recalling that the angle θ between $x - x_0$ and $y - x$ is determined by

$$(x - x_0, y - x) = \cos(\theta) \|x - x_0\| \|y - x\|,$$

we see (2.5) means x is that point of K for which $-\pi/2 \leq \theta \leq \pi/2$ for every $y \in K$. We define x to be the *projection* of x_0 on K and denote it by $P_K(x_0)$.



COROLLARY 2.1. *For each closed convex non-empty subset K of V there is a projection operator $P_K : V \rightarrow K$ for which $P_K(x_0)$ is that point of K closest to $x_0 \in V$; it is characterized by*

$$P_K(x_0) \in K : (P_K(x_0) - x_0, y - P_K(x_0)) \geq 0, \quad y \in K.$$

It follows from this characterization that the function P_K satisfies

$$\|P_K(x_0) - P_K(y_0)\|^2 \leq (P_K(x_0) - P_K(y_0), x_0 - y_0), \quad x_0, y_0 \in V.$$

From this we see that P_K is a *contraction*, i.e.,

$$\|P_K(x_0) - P_K(y_0)\| \leq \|x_0 - y_0\|, \quad x_0, y_0 \in V,$$

and that P_K satisfies the *angle condition*

$$(P_K(x_0) - P_K(y_0), x_0 - y_0) \geq 0, \quad x_0, y_0 \in V.$$

That is, the operator P_K is *monotone* (cf., Section II.2).

COROLLARY 2.2. *For each closed subspace K of V and each $x_0 \in V$ there is a unique*

$$x \in K : (x - x_0, y) = 0, \quad y \in K.$$

Two vectors $x, y \in V$ are called *orthogonal* if $(x, y) = 0$, and the *orthogonal complement* of the set S is $S^\perp \equiv \{x \in V : (x, y) = 0 \text{ for } y \in S\}$. Corollary 2.2 says each $x_0 \in V$ can be uniquely written in the form $x_0 = x_1 + x_2$ with $x_1 \in K$ and $x_2 \in K^\perp$ whenever K is a closed subspace. We denote this orthogonal decomposition by $V = K \oplus K^\perp$.

The *Riesz map* $\mathcal{R}$ of V into V' is defined by $\mathcal{R}x(y) = (x, y)$ for $x, y \in V$. It is clear that $\|\mathcal{R}x\|_{V'} = \|x\|_V$; Theorem 1 with $K = V$ shows by way of (2.5) that $\mathcal{R}$ is onto V' , so $\mathcal{R}$ is an isometric isomorphism of the Hilbert space V onto its dual V' . Specifically, for each $f \in V'$ there is a unique $x = \mathcal{R}^{-1}(f) \in V$.

COROLLARY 2.3. *For each $f \in V'$ there is a unique*

$$(2.6) \quad x \in V : (x, y) = f(y) , \quad y \in V .$$

We recognize (2.6) as the weak formulation of certain boundary value problems. Specifically, when $V = H_0^1$ or H^1 , (2.6) is the Dirichlet or Neumann problem, respectively, with $c = 1$. An easy but useful generalization is obtained as follows. Let $a : V \times V \rightarrow \mathbb{R}$ be bilinear (linear in each variable separately), continuous (cf. (1.1)), symmetric ($a(x, y) = a(y, x)$, $x, y \in V$) and *V-elliptic*: there is a $c_0 > 0$ such that

$$(2.7) \quad a(x, x) \geq c_0 \|x\|^2 , \quad x \in V .$$

Thus, $a(\cdot, \cdot)$ determines an *equivalent scalar product* on V : a sequence converges in V with $\|\cdot\|$ if and only if it converges with $a(\cdot, \cdot)^{1/2}$. Thus we may replace $(\cdot, \cdot)$ by $a(\cdot, \cdot)$ above.

THEOREM 2.1A. *Let $a(\cdot, \cdot)$ be a bilinear, symmetric, continuous and V-elliptic form on the Hilbert space V , let K be a closed, convex and non-empty subset of V , and let $f \in V'$. Set $\varphi(x) = (1/2)a(x, x) - f(x)$, $x \in V$. Then there is a unique*

$$(2.8) \quad x \in K : \varphi(x) \leq \varphi(y) , \quad y \in K .$$

The solution of (2.8) is characterized by

$$(2.9) \quad x \in K : a(x, y - x) \geq f(y - x) , \quad y \in K .$$

If, in addition, K is a subspace of V , then (2.9) is equivalent to

$$(2.10) \quad x \in K : a(x, y) = f(y) , \quad y \in K .$$

Now (2.10) is precisely our weak formulation, and we see it is the special case of a *variational inequality* (2.9) which is the characterization of the solution of the minimization problem (2.8). When $a(\cdot, \cdot)$ is not symmetric we can still solve the linear problem (2.10), although it no longer is related to a minimization problem.

THEOREM 2.2 (LAX-MILGRAM). *Let $a(\cdot, \cdot)$ be bilinear, continuous and V-coercive (see 1.2), and let $f \in V'$. Then there is a unique*

$$(2.11) \quad x \in V : a(x, y) = f(y) , \quad y \in V .$$

PROOF. For each $x \in V$ the function " $y \mapsto a(x, y)$ " belongs to V' , so by Corollary 3 there is a unique $\alpha(x) \in V : (\alpha(x), y) = a(x, y)$, $y \in V$. This defines $\alpha \in \mathcal{L}(V, V)$, and we similarly construct $\beta \in \mathcal{L}(V, V)$ with $(x, \beta(y)) = a(x, y)$ for $x, y \in V$. Since (2.11) is equivalent to $\alpha(x) = \mathcal{R}^{-1}(f)$, it suffices to show α is invertible. First, α is one-to-one, since

$$c_0 \|x\|^2 \leq |a(x, x)| = |(\alpha(x), x)| \leq \|\alpha(x)\| \|x\| ,$$

and so $\alpha(x) = 0$ implies $x = 0$. Also, $c_0 \|x\| \leq \|\alpha(x)\|$ for all $x \in V$. Second, we show the range of α , $Rg(\alpha)$, is closed. If $\lim_{n \rightarrow \infty} z_n = z$ and $z_n = \alpha(x_n)$, then

$c_0\|x_n - x_m\| \leq \|z_n - z_m\|$ so $\{x_n\}$ is Cauchy, hence, convergent to some $x \in V$. But α is continuous, so $\alpha(x) = z \in Rg(\alpha)$. Finally, since $K \equiv Rg(\alpha)$ is a closed subspace, hence $V = Rg(\alpha) \oplus Rg(\alpha)^\perp$, we need only show $Rg(\alpha)^\perp = \{0\}$. But if $y \in Rg(\alpha)^\perp$ then for every $x \in V$, $0 = (\alpha(x), y) = (x, \beta(y))$, so $\beta(y) = 0$. As above, β is one-to-one, so $y = 0$. Thus $Rg(\alpha) = V$. $\square$

Let's consider briefly the dependence of the solution of (2.11) on the data $a(\cdot, \cdot)$ and f in the problem. We denote by $\mathcal{A}$ the operator in $\mathcal{L}(V, V')$ which is equivalent to the bilinear form $a(\cdot, \cdot)$ and given by

$$\mathcal{A}x(y) = a(x, y), \quad x, y \in V.$$

Thus (2.11) is equivalent to $\mathcal{A}x = f$, and we note that $\mathcal{A} = \mathcal{R} \circ \alpha$, where α arose in the proof of Theorem 2. Suppose we are given two such bilinear forms and linear functionals. Denote the corresponding operators by $\mathcal{A}_1$, $\mathcal{A}_2$, and consider the solutions of $\mathcal{A}_1(x_1) = f_1$, $\mathcal{A}_2(x_2) = f_2$. Then we have

$$\mathcal{A}_1(x_1 - x_2)(x_1 - x_2) = (f_1 - f_2)(x_1 - x_2) + (\mathcal{A}_2 - \mathcal{A}_1)x_2(x_1 - x_2),$$

and from here we obtain

$$c_0\|x_1 - x_2\|^2 \leq (\|f_1 - f_2\|_{V'} + \|(\mathcal{A}_2 - \mathcal{A}_1)x_2\|_{V'})\|x_1 - x_2\|,$$

where c_0 is the constant in (2.7) for $\mathcal{A}_1$. This yields in turn the estimate

$$c_0\|x_1 - x_2\| \leq \|f_1 - f_2\|_{V'} + \|\mathcal{A}_2 - \mathcal{A}_1\|_{\mathcal{L}(V, V')}\|f_2\|_{V'}/c_0.$$

COROLLARY 2.4. *Assume, in addition, that $a(\cdot, \cdot)$ is V -elliptic. Then the solution of (2.11) in V depends continuously on f in V' and on $\mathcal{A}$ in $\mathcal{L}(V, V')$.*

Finally we show that the nonlinear problem (2.9) can be resolved for non-symmetric forms. The proof will make use of the following elementary fixed-point theorem.

PROPOSITION 2.3. *Let K be a closed non-empty subset of a Banach space V , and let $T : K \rightarrow K$ be a strict contraction:*

$$\|T(x) - T(y)\| \leq \lambda\|x - y\|, \quad x, y \in V,$$

where $0 \leq \lambda < 1$. Then T has a unique fixed point, an $x \in K : T(x) = x$.

PROOF. Let $x_0 \in K$ and define $\{x_n\}$ by $T(x_n) = x_{n+1}$, $n \geq 0$. For $n, k \geq 1$ we have

$$\begin{aligned} \|x_{n+k+1} - x_n\| &\leq \sum_{j=n}^{n+k} \|x_{j+1} - x_j\| \\ &\leq \sum_{j=n}^{n+k} \lambda^j \|x_1 - x_0\| \\ &\leq \lambda^n (1 - \lambda)^{-1} \|x_1 - x_0\|, \end{aligned}$$

so $\{x_n\}$ is a Cauchy sequence in K . The limit of this sequence is a fixed point in K . Finally, if x_1 and x_2 are fixed points, $\|x_1 - x_2\| = \|T(x_1) - T(x_2)\| \leq \lambda\|x_1 - x_2\|$, so $(1 - \lambda)\|x_1 - x_2\| = 0$. Thus, $x_1 = x_2$, so there is exactly one fixed point of T . $\square$

COROLLARY 2.5. *Let K and V be as above and assume T^n is a strict-contraction for some integer $n \geq 1$. Then T has a unique fixed point in K .*

THEOREM 2.3 (LIONS-STAMPACCHIA). *Let $a(\cdot, \cdot)$ be a bilinear, continuous and V -elliptic form on V , and let K be a closed, convex and nonempty subset of V . Then for each $f \in V'$ there exists a unique*

$$(2.12) \quad x \in K : a(x, y - x) \geq f(y - x), \quad y \in K,$$

and the mapping $f \mapsto x : V' \mapsto K$ is continuous.

PROOF. Let x_1 and x_2 be solutions corresponding to f_1 and f_2 . Then $a(x_1, x_2 - x_1) \geq f_1(x_2 - x_1)$, $a(x_2, x_1 - x_2) \geq f_2(x_1 - x_2)$, and we add these to get $a(x_1 - x_2, x_1 - x_2) \leq (f_1 - f_2)(x_1 - x_2)$. This gives $\|x_1 - x_2\| \leq (1/c_0)\|f_1 - f_2\|_{V'}$ from which follows the uniqueness and continuous dependence.

To prove existence, let $r > 0$ and define $F(x) \in V'$ for each $x \in V$ by

$$F(x)(y) = (x, y) - ra(x, y) + rf(y), \quad y \in V.$$

Then note that x is a solution (2.12) if and only if

$$x \in K : (x, y - x) \geq F(x)(y - x), \quad y \in K.$$

But this is equivalent to $x = P_K(\mathcal{R}^{-1}F(x))$; so x is characterized as the fixed point of the function $P_K\mathcal{R}^{-1}F$. Now P_K is a contraction, and $\mathcal{R}$ is an isometric isomorphism, so it suffices to show F is a strict contraction. But we have

$$|(F(x_1) - F(x_2))(y)| = |(x_1 - x_2, y) - r(\alpha(x_1 - x_2), y)|$$

where $\alpha : V \rightarrow V$ was constructed in Theorem 2.2, and

$$\|x - r\alpha(x)\|^2 = \|x\|^2 - 2ra(x, x) + r^2\|\alpha(x)\|^2 \leq (1 - 2rc_0 + r^2C^2)\|x\|^2.$$

Choose $r < 2c_0/C^2$ so $\lambda \equiv (1 - 2rc_0 + r^2C^2)^{1/2} < 1$. Then we have $\|F(x_1) - F(x_2)\|_{V'} \leq \lambda\|x_1 - x_2\|$, so it follows that $P_K\mathcal{R}^{-1}F$ has a unique fixed point. $\square$

In the following section we shall illustrate how the Theorems of this section can be applied to the Sobolev spaces constructed in Section 1 to show that the boundary-value problems of Dirichlet, Neumann and various other types are all well-posed. Extensions of these Theorems to considerably more general situations and related nonlinear boundary-value problems in $\mathbb{R}^n$ will be developed in Chapter II. Also see Theorem III.2.1 for an extension of Theorem 2.2 which is applicable to linear evolution equations.

I.3. Applications to Stretched String Problems

We consider the simplest problem of elasticity, the vertical displacement $u(x)$ within a fixed plane of a string of length $\ell > 0$ whose initial position ($u = 0$) is the interval $0 \leq x \leq \ell$. The string is stretched by a tension $T > 0$ and it is flexible and elastic, so this tension acts in the direction of the tangent and the string has no resistance to bending. A vertical load or force $F(x)$ per unit length is applied and this results in the displacement $u(x)$ at each point x . For each segment $[x_1, x_2]$ the vertical components of force must balance, and this gives

$$-T \sin \theta_{x_2} + T \sin \theta_{x_1} = \int_{x_1}^{x_2} F(x) dx$$

where $\sin \theta_x = u'(x)/\sqrt{1+(u'(x))^2}$ is the vertical component of the unit tangent at $(x, u(x))$. We assume displacements are small, so $1+(u')^2 \cong 1$ and we obtain

$$-T(u'(x_2) - u'(x_1)) = \int_{x_1}^{x_2} F(x) dx, \quad 0 \leq x_1 < x_2 \leq \ell.$$

If F is locally integrable on $(0, \ell)$ it follows that $u' = \partial u$ is locally absolutely continuous and the equation

$$(3.1) \quad -T\partial^2 u = F \quad \text{in } L^1_{\text{loc}}(0, \ell)$$

describes the displacement of the string in the interior of the interval $(0, \ell)$. At the end-points we need to separately prescribe the behavior. For example, at $x = 0$ we could specify either the position, $u(0) = c$, the vertical force, $T\partial u(0) = f_0$, or some combination of these such as an elastic restoring force of the form $T\partial u(0) = h(u(0) - c)$. Such conditions will be prescribed at each of the two boundary points of the interval.

To calculate the energy that is added to the string to move it to the position u , we take the product of the forces and displacements. These tangential and vertical changes are given by

$$T \int_0^\ell (\sqrt{1+(u')^2} - 1) dx - \int_0^\ell F(x)u(x) dx$$

where the first term depends on the change in length of the string, and for small displacements this gives us the approximate *potential energy* functional

$$(3.2) \quad \varphi(u) = \int_0^\ell ((T/2)(\partial u)^2 - Fu) dx.$$

Here we have used the expansion for small values of r

$$\sqrt{1+r^2} = 1 + \frac{1}{2}r^2 + \dots$$

We shall see the displacement u corresponding to the external load F can be obtained by minimizing (3.2) over the appropriate set of admissible displacements. Moreover, this applies to more general loads. For example, a “point load” of magnitude $F_0 > 0$ applied at the point c , $0 < c < \ell$, leads to the energy functional

$$\varphi(u) = T/2 \int_0^\ell (\partial u)^2 dx - F_0 u(c)$$

in which the second term is just a Dirac functional concentrated at c .

Next we give a set of boundary-value problems on the interval (a, b) . Each is guaranteed to have a unique solution by Theorem 2.1A. Each example will be related to a stretched-string problem, and for certain special cases we shall compute the displacement u to see if it appears to be consistent with the physical problem. In all of these examples, we rescale the load so that we may assume $T = 1$. Thus, the load becomes the ratio of the actual load to the tension.

EXAMPLE 3.A. The displacement of a string fixed at both ends is given by the solution of

$$u \in H_0^1(0, \ell) : -\partial^2 u = f$$

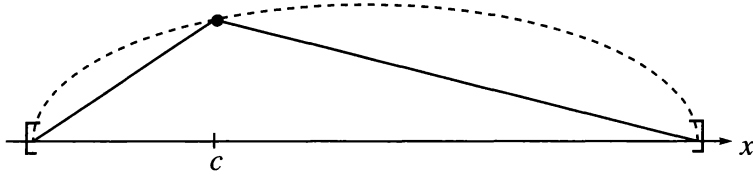
where $f \in H_0^1(0, \ell)'$. This problem is well-posed by Theorem 2.1A; note that the corresponding bilinear form is H_0^1 -elliptic by the estimate (1.4). If we apply a load $F(x) = \text{sgn}(x - \ell/2)$, where the *sign function* is given by $\text{sgn}(x) = x/|x|$, $x \neq 0$, the resulting displacement is

$$u(x) = \begin{cases} -\frac{1}{2}x(\ell/2 - x), & 0 < x < \ell/2 \\ \frac{1}{2}(x - \ell/2)(\ell - x), & \ell/2 < x < \ell \end{cases}$$



If we apply a point load, δ_c concentrated at $x = c$, the displacement is

$$u(x) = 1/2(c - |x - c|) + (1/2 - c/\ell)x$$



with maximum value $u(c) = c(1 - c/\ell)$. Both of these solutions can be computed directly from the ordinary differential equation by using Proposition 1.2.

EXAMPLE 3.B. Non-homogeneous boundary conditions arise when the displacements at the end-points are fixed at non-zero levels. For example, the solution to

$$u \in H^1(0, \ell) : u(0) = f_1, \quad u(\ell) = f_2, \quad -\partial^2 u = F$$

is obtained by minimizing (3.2) over the set of admissible displacements

$$K = \{v \in H^1(0, \ell) : v(0) = f_1, v(\ell) = f_2\}.$$

This minimum u satisfies (2.9) where

$$a(u, v) = \int_0^\ell \partial u \partial v \, dx, \quad f(v) = \int_0^\ell Fv \, dx \quad u, v \in H^1(0, \ell).$$

Since the set K is the translate of the subspace $H_0^1(0, \ell)$ by the function

$$u_0(x) = (\ell - x)f_1/\ell + xf_2/\ell,$$

this variational inequality is equivalent to

$$u \in K : a(u, \varphi) = f(\varphi), \quad \varphi \in H_0^1(0, \ell).$$

Moreover, this problem is actually a “linear” problem for the unknown $w \equiv u - u_0$ in the form

$$w \in H_0^1(0, \ell) : a(w, \varphi) = f(\varphi) - a(u_0, \varphi), \quad \varphi \in H_0^1(0, \ell),$$

and thus it is well-posed by any one of the Theorems of Section 2.

EXAMPLE 3.C. Unilateral boundary constraints arise when the admissible displacements are given by

$$K = \{v \in H^1(0, \ell) : v(0) = 0, \quad v(\ell) \geq 0\}.$$

Since K is a *cone*, the displacement u that minimizes (3.2) over K is characterized by

$$\begin{aligned} u \in K : a(u, \varphi) &\geq f(\varphi), \quad \text{for } \varphi \in H^1(0, \ell), \quad \varphi(0) = 0, \quad \varphi(\ell) \geq 0, \\ \text{and } a(u, u) &= f(u), \end{aligned}$$

and this is equivalent to

$$\begin{aligned} u \in H^1(0, \ell) : -\partial^2 u &= F \quad \text{in } L^2(0, \ell), \\ u(0) &= 0, \quad \text{and} \quad u(\ell) \geq 0, \quad \partial u(\ell) \geq 0, \quad \partial u(\ell)u(\ell) = 0. \end{aligned}$$

The conditions at ℓ state that the displacement is non-negative, the vertical force of the constraint is non-negative, and one or the other is equal to zero. For the problem in which a force is loaded at the point c , $0 < c < \ell$, with magnitude F_0 , the functional is

$$f(v) = F_0 v(c) = F_0 \delta_c(v), \quad v \in H^1(0, \ell),$$

and the corresponding solutions can be computed as follows. The general solution of

$$-\partial^2 u = F_0 \delta_c, \quad u(0) = 0$$

is given by

$$u(x) = c_1 x - F_0(x - c)H(x - c), \quad 0 \leq x \leq \ell,$$

and we need additionally to have

$$\begin{aligned} u(\ell) &= c_1 \ell - F_0(\ell - c) \geq 0, \\ \partial u(\ell) &= c_1 - F_0 \geq 0 \end{aligned}$$

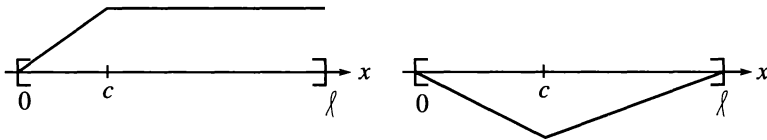
with equality in at least one. Thus we have either

$$c_1 = F_0 \geq 0 : u(x) = F_0(x - (x - c)H(x - c)),$$

or else

$$c_1 = (1 - c/\ell)F_0 \leq 0 : u(x) = F_0((1 - c/\ell)x - (x - c)H(x - c))$$

depending on the sign of F_0 . Thus, if $F_0 \geq 0$



the string does not touch the constraint at $x = \ell$ and has zero vertical force acting on that end, while if $F_0 < 0$ the string is supported by the constraint with a force given by $\partial u(\ell) = -(c/\ell)F_0 > 0$.

EXAMPLE 3.D. Unilateral internal constraints on the string occur, for example, if the entire string is constrained as for the admissible displacements

$$K = \{v \in H_0^1(0, \ell) : v(x) \geq 0, 0 \leq x \leq \ell\}.$$

With a distributed load $F(x)$ as above the solution is characterized by

$$(3.3) \quad u \in H_0^1(0, \ell) : u \geq 0, \quad \partial^2 u + F \leq 0, \quad (\partial^2 u + F)(u) = 0.$$

For the specific load given by $F(x) = \text{sgn}(x - \ell/2)$ we can directly compute a strong solution of this problem, and it is given by

$$u(x) = \begin{cases} \frac{1}{2}\left(\frac{\ell}{2}\right)^2 + (\sqrt{2} - 1)\left(\frac{\ell}{2}\right)(x - \ell) - \frac{1}{2}(x - \ell/2)^2 \text{sgn}(x - \ell/2), & (1 - \sqrt{2}/2)\ell \leq x \leq \ell, \\ 0, & 0 \leq x \leq (1 - \sqrt{2}/2)\ell. \end{cases}$$

Thus we find the interval is divided into two distinct regions. In the first, the displacement is up against the obstacle or constraint, $u = 0$. In the other, the displacement is free of the constraint and it satisfies the differential equation. These two alternatives are implied by the equation in (3.3). The first inequality is the constraint and the second states that the effect of the constraint is that of a non-negative force, $-(\partial^2 u + F)$, distributed along the string. Note that the dependence of the displacement u on the force F is non-linear. The location of the first region of constraint is the major difficulty in this problem, for if it were known then one could obtain the solution by resolving a Dirichlet problem on the complementary region. This is an example of a *free-boundary problem* wherein the crux is to locate the unknown boundary between the contact region and its complement.

EXAMPLE 3.E. Our next problem arises from a stretched string which has a specified vertical force at the left end and an elastic restoring force at the right end. Thus, define $V = H^1(0, \ell)$ and

$$\begin{aligned} a(u, v) &= \int_0^\ell \partial u \partial v \, dx + hu(\ell)v(\ell), \quad u, v \in V, \\ f(v) &= \int_0^\ell Fv \, dx - f_0v(0) + f_\ell v(\ell), \quad v \in V. \end{aligned}$$

From (1.5) we find these are continuous functionals on V , and by similar estimates it follows that $a(\cdot, \cdot)$ is $H^1(0, \ell)$ -elliptic if $h > 0$. With this assumption, it follows from Theorem 2.1A that there is a unique weak solution of the boundary-value problem

$$\begin{aligned} u &\in H^1(0, \ell) : -\partial^2 u = F \quad \text{in } L^2(0, \ell), \\ \partial u(0) &= f_0, \quad \partial u(\ell) + hu(\ell) = f_\ell. \end{aligned}$$

Note that the equation in $L^2(0, \ell)$ shows that ∂u is absolutely continuous on $[0, \ell]$, so the boundary conditions are meaningful. Finally, we see that $a(\cdot, \cdot)$ fails to be H^1 -elliptic if $h = 0$. In that case there is non-uniqueness of solutions, since any constant added to a solution gives another solution, and there exists a solution only if

$$\int_0^\ell F(x) \, dx - f_0 + f_\ell = 0,$$

i.e., the total force on the string is zero. In the absence of a restoring force ($h = 0$) this constraint on the forces is necessary for existence of a physically consistent displacement of the stretched string.

Orientation.

We saw in Section 1 that various boundary-value problems have a weak formulation in Sobolev spaces which are actually equivalent to the respective strong or direct formulations. Then in Section 2 we found that this weak formulation is equivalent to a minimization principle (2.2), and that it corresponds to a variational problem, (2.5) or (2.11). Section 3 indicates that these can be related to “minimum energy” statements in a classical setting.

Linear boundary-value-problems can be characterized by operators on Hilbert space in two rather natural but different ways. The first is motivated by the abstract variational problem (2.11). Each bilinear continuous form $a(\cdot, \cdot)$ on the Hilbert space V is equivalent to a continuous linear operator $\mathcal{A} : V \rightarrow V'$, and these are related by

$$a(x, y) = \mathcal{A}x(y) , \quad x, y \in V .$$

The problem (2.11) is equivalent to the operator equation

$$x \in V : \mathcal{A}x = f \text{ in } V' .$$

This operator $\mathcal{A}$ from the space to its dual is one-to-one or onto exactly when the boundary-value problem has uniqueness or is always solvable, respectively, and $\mathcal{A}^{-1}$ being continuous (e.g., $a(\cdot, \cdot)$ is V -coercive) corresponds to the continuous dependence of the solution x on the data f in (2.11).

The second approach is to characterize the boundary-value problem as an *unbounded* operator A on a single space H : the domain $D(A)$ of the operator A is the set of functions in H which satisfy all the boundary conditions, and the value of A is just the action of the differential equation on any such function in $D(A)$. Since the domain and range both are in the same space, various polynomials (or more general functions) of A or its *resolvent*, $(\lambda I + A)^{-1}$, can be constructed in H . However, such an operator is almost never continuous, and the domain is usually a proper subset of the Hilbert space, so the analysis is rather delicate. We shall briefly study this second way of characterizing boundary-value problems and its connection with the corresponding variational operators $\mathcal{A}$. Then we study in Section 5 the exponential function of such an unbounded operator A which characterizes the corresponding Cauchy problem.

I.4. Unbounded Operators

Let H be a Hilbert space, D a subspace (algebraic) of H and let $A : D \rightarrow H$ be linear. Such a map we call an *unbounded operator* on H with *domain* D . The *graph* of A is the subspace

$$G(A) = \{[x, Ax] : x \in D\}$$

of the product $H \times H$. Note that $H \times H$ is also a Hilbert space with componentwise addition and scalar multiplication, and its scalar product is

$$([x_1, x_2], [y_1, y_2])_{H \times H} = (x_1, y_1)_H + (x_2, y_2)_H .$$

The operator A is called *closed* if $G(A)$ is a closed subspace of $H \times H$. That is, A is closed if whenever $x_n \in D$, $x_n \rightarrow x$ and $Ax_n \rightarrow y$ in H imply that $x \in D$ and

$Ax = y$. This is a much weaker condition than continuity of A , since convergence of $\{Ax_n\}$ is an assumption, not a conclusion.

Suppose $A : D \rightarrow H$ is an unbounded operator on H and that D is *dense* in H . The *adjoint* of A is defined as follows. Let D^* be the subspace of those $y \in H$ for which the linear map $x \mapsto (Ax, y) : D \rightarrow \mathbb{R}$ is continuous. Such a map has a unique extension by uniform continuity to all of H and thus there is a unique vector $A^*y \in H$ such that

$$(4.1) \quad (Ax, y)_H = (x, A^*y)_H, \quad x \in D, y \in D^*.$$

It follows easily that $A^* : D^* \rightarrow H$ is an unbounded operator (i.e., linear) on H , and it is called the *adjoint* of A .

Since A^* is defined by (4.1) with D^* being “maximal”, it follows directly that A^* is closed. Note that we can define A^* only if D is dense.

LEMMA 4.1. *If A is closed and D is dense, then D^* is dense.*

PROOF. Let $P : H \times H \rightarrow G(A)^\perp$ be the projection onto the orthogonal complement of $G(A)$ in $H \times H$. If $y \in H$, $y \neq 0$, then $[0, y] \notin G(A)$ so $P[0, y] = [u, v]$ satisfies $([u, v], [0, y])_{H \times H} = (v, y)_H \neq 0$. But

$$(u, x)_H + (v, Ax)_H = 0, \quad x \in D$$

so $v \in D^*$. Thus, we see that for each $y \neq 0$ there is a $v \in D^*$ with $(v, y)_H \neq 0$. This shows $(D^*)^\perp = \{0\}$, so D^* is dense. $\square$

DEFINITION. An (unbounded) operator $A : D \rightarrow H$ is *accretive* if

$$(Ax, x)_H \geq 0, \quad x \in D,$$

and it is *m-accretive* if, in addition, $A + I$ maps D onto H , i.e., $Rg(A + I) = H$.

Such operators will occur frequently and play an important role in our work below. We give some examples in $H = L^2(a, b)$.

EXAMPLE 4.A. Set $D = H_0^1(a, b)$ and $A = \partial$. Then A is closed, D is dense, and the adjoint is given by $A^* = -\partial$ on $D^* = H^1(a, b)$. Note that A is accretive, A^* is not accretive, and that $Rg(A + I)$ is the orthogonal complement of $\{\exp(x)\}$. Thus A is not *m-accretive*. Finally, the second adjoint A^{**} is equal to A ; in particular, it has the same domain.

EXAMPLE 4.B. Set $D = \{v \in H^1(a, b) : v(a) = cv(b)\}$ and $A = \partial$. Then $A^* = -\partial$ on $D^* = \{v \in H^1(a, b) : v(b) = cv(a)\}$ and A is closed, as follows directly or by comparison with A^* . Furthermore, A and A^* are accretive only if $|c| \leq 1$, and then both are *m-accretive* with

$$(I + A)u = f \quad \text{if and only if} \\ u(x) = \int_a^b G(x, s)f(s) ds, \quad a \leq x \leq b,$$

where

$$G(x, s) = \frac{e^{-(x-s)}}{e^{b-a} - c} \begin{cases} e^{b-a}, & a \leq s < x, \\ c, & x < s \leq b. \end{cases}$$

The integrand $G(\cdot, \cdot)$ is the Green's function for $A + I$, and it is characterized for each $s \in (a, b)$ as the solution of

$$G(\cdot, s) \in D, \quad (I + A)G(\cdot, s) = \delta_s,$$

where δ_s is the Dirac functional at s .

EXAMPLE 4.C. Set $D = \{v \in H_0^1(a, b) : \partial^2 v \in L^2(a, b)\}$ and $A = -\partial^2$. Then one can show directly that A is closed, or that $A = A^*$, from which it follows that A is closed. It is easy to check that A is accretive and from Section 2 that A is m -accretive. This operator corresponds to the *Dirichlet problem*.

LEMMA 4.2. *If A is m -accretive then $(I + A)^{-1}$ is a contraction on H , hence, A is closed.*

PROOF. For $u \in D$ we have

$$\|u\|_H^2 \leq ((I + A)u, u)_H \leq \|(I + A)u\|_H \|u\|_H$$

so $(I + A)^{-1}$ is a contraction. Since an operator is closed if and only if its inverse is closed, and it is closed if and only if its sum with a multiple of the identity is closed, the result follows. $\square$

Actually Example 4.C illustrates a general situation that occurs frequently. Let V be a Hilbert space which is dense in another Hilbert space H , and assume the identity $V \rightarrow H$ is continuous. Let $a(\cdot, \cdot)$ be a continuous bilinear form on V . Then we define D to be the set of all $u \in V$ such that the function $v \mapsto a(u, v)$ is continuous on V with the H -norm. For each such $u \in D$ there is then a unique $Au \in H$ such that

$$a(u, v) = (Au, v)_H, \quad u \in D, v \in V,$$

and this defines a linear operator $A : D \rightarrow H$. This construction is similar to that of the adjoint above, and the special case of

$$H = L^2(a, b), \quad V = H_0^1(a, b), \quad a(u, v) = (\partial u, \partial v)_H$$

gives our last example.

Consider the *adjoint form* on V given by $b(u, v) = a(v, u)$, $u, v \in V$. This leads likewise to an operator $B : D(B) \rightarrow H$ given by

$$a(u, v) = (u, Bv)_H, \quad u \in V, v \in D(B).$$

Then we obtain the following.

PROPOSITION 4.1. *Assume there is a $\lambda \in \mathbb{R}$ and a $c > 0$ such that*

$$a(v, v) + \lambda \|v\|_H^2 \geq c \|v\|_V^2, \quad v \in V.$$

Then D is dense in H , the operator $A + \lambda I : D(A) \rightarrow H$ is one-to-one and onto and its inverse is continuous, A is closed, and $A^ = B$.*

PROOF. If $F \in H$ then $v \mapsto (F, v)_H$ is continuous and linear on V , so by Theorem 2.2 there is a unique

$$u \in V : a(u, v) + \lambda(u, v)_H = (F, v)_H, \quad v \in V.$$

Thus $u \in D$ and $(A + \lambda)u = F$, so $A + \lambda$ maps D one-to-one onto H . Similarly, $B + \lambda$ maps $D(B)$ one-to-one onto H . If $w \in D^\perp$ in H then there is a $v \in D(B)$ with $(B + \lambda)v = w$, and so

$$0 = (u, w)_H = (u, (B + \lambda)v)_H = ((A + \lambda)u, v)_H, \quad u \in D.$$

Since $A + \lambda I$ is onto, $v = 0$ and $w = 0$, so $D^\perp = \{0\}$. This shows D is dense. We can deduce that A is closed from the fact that $(A + \lambda)^{-1}$ is continuous or from $A = B^*$, which follows by symmetry from the next part of the proof.

Suppose $v \in D(B)$. Then for every $u \in D$ we have $(Au, v)_H = a(u, v) = (u, Bv)$, and this shows that $v \in D(A^*)$ with $A^*v = Bv$. That is, A^* is an extension of B . Next let $u \in D^*$ and choose $u_0 \in D(B)$ so that $(B + \lambda)u_0 = (A^* + \lambda)u$. For each $v \in D$ we have

$$((A + \lambda)v, u)_H = (v, (A^* + \lambda)u)_H = (v, (B + \lambda)u_0)_H = ((A + \lambda)v, u_0)_H.$$

Since $A + \lambda$ is onto H this implies $u = u_0 \in D(B)$, hence, $D^* = D(B)$. $\square$

COROLLARY 4.1. Assume that $a(\cdot, \cdot)$ is non-negative, i.e.,

$$a(v, v) \geq 0, \quad v \in V,$$

and that there is a $c > 0$ for which

$$a(v, v) + \|v\|_H^2 \geq c\|v\|_V^2, \quad v \in V.$$

Then A is m -accretive.

Here is another example to illustrate the situation, the *Neumann problem*.

EXAMPLE 4.D. Set $H = L^2(a, b)$, $V = H^1(a, b)$ and define

$$a(u, v) = \int_a^b \partial u \partial v, \quad u, v \in V.$$

As above this determines an unbounded operator A on $L^2(a, b) : Au = F \in L^2(a, b)$ is equivalent to

$$u \in V : a(u, v) = (F, v)_{L^2}, \quad v \in V,$$

and this weak Neumann problem is equivalent to

$$u \in V, \quad -\partial^2 u = F, \quad \partial u(a) = \partial u(b) = 0.$$

Thus, we find

$$D = \{u \in V : \partial^2 u \in L^2(a, b), \partial u(a) = \partial u(b) = 0\}$$

and $A = -\partial^2 : D \rightarrow L^2(a, b)$. From Proposition 1 it follows that A is m -accretive, $A = A^*$, and that $A + \lambda I$ is a bijection of D onto $L^2(a, b)$ for every $\lambda > 0$. The situation is different at $\lambda = 0$. Specifically, it is clear that $Au = F$ is possible only if $\int_a^b F = 0$, i.e., F is orthogonal in L^2 to the constant functions. Conversely, one

can show that this condition on F is sufficient for the existence of a solution. (For example, solve the mixed problem

$$-\partial^2 u = F, \quad u(a) = \partial u(b) = 0,$$

and then note that $\int_a^b F = \partial u(a)$.) In summary, we find the range of A and kernel of A are given by

$$\text{Ker}(A) = \{ \text{constant functions} \},$$

$$\text{Rg}(A) = \text{Ker}(A)^\perp.$$

We shall characterize the m -accretive operators among those which are accretive. But first, note that if $B \in \mathcal{L}(H)$ is a strict contraction, $\|B\|_{\mathcal{L}(H)} < 1$, then $I - B$ is a bijection of H onto itself. This follows directly, either from the power-series representation $(I - B)^{-1} = \sum_{n=0}^{\infty} B^n$ in $\mathcal{L}(H)$, from Proposition 2.3, or from Proposition 4.1 with $V = H$ and $a(u, v) = ((I - B)u, v)_H$.

PROPOSITION 4.2. *The following are equivalent:*

- (a) $A : D \rightarrow H$ is accretive and there exists a $\mu > 0$ such that $\text{Rg}(\mu I + A) = H$,
- (b) A is m -accretive, and
- (c) A is accretive, D is dense in H , and $\text{Rg}(\lambda I + A) = H$ for every $\lambda > 0$.

PROOF. Clearly, (c) implies (b) and (b) implies (a). Suppose (a) holds. Then $(\mu + A)^{-1} \in \mathcal{L}(H)$ and $\|(\mu + A)^{-1}\| \leq 1/\mu$. From our preceding remark it follows that if $|\lambda - \mu|/\mu < 1$, then $(I + (\lambda - \mu)(\mu + A)^{-1})^{-1} \in \mathcal{L}(H)$. In this case we have

$$(\lambda + A)(\mu + A)^{-1} = ((\lambda - \mu) + (\mu + A))(\mu + A)^{-1} = ((\lambda - \mu)(\mu + A)^{-1} + I),$$

so $\text{Rg}(\lambda + A) = H$ whenever $0 < \lambda < 2\mu$. By induction it follows that $\text{Rg}(\lambda + A) = H$ for every $\lambda > 0$. To show D is dense, let $z \in D^\perp$, set $z = (\mu + A)x$ for some $x \in D$, and note that $0 = (z, x)_H \geq \mu\|x\|^2$, so $x = 0$, hence, $z = 0$. This shows $D^\perp = \{0\}$. The preceding shows (a) implies (c), and so the equivalences are now obvious. $\square$

In the next section we shall show that the m -accretive operators are further characterized as those for which the *initial-value problem*

$$\frac{du(t)}{dt} + Au(t) = 0, \quad 0 < t, \quad u(0) = u_0$$

is a well-posed problem. Formally, we write $u(t) \cong \exp(-tA)u_0$; the m -accretive operators are precisely those for which the indicated “exponential operators” can be constructed as contractions on H .

It is easy to see how the unbounded operator A with domain D in H constructed as above from the continuous bilinear form $a(\cdot, \cdot)$ on V is related to the continuous $\mathcal{A} \in \mathcal{L}(V, V')$ which is equivalent to $a(\cdot, \cdot)$. In fact, the graph of A is the restriction of the graph of $\mathcal{A}$ to $V \times H$. That is, note that $H' \hookrightarrow V'$ by “restriction to V ” of functionals on H , so $D = \{u \in V : \mathcal{A}u \in H'\}$ and then $Au \in H$ is just that $\mathcal{A}u \in H'$ which corresponds through the identification of H with H' by its Riesz map. Thus, with this identification $H = H'$ in the proof of Proposition 4.1, it is clear that $\mathcal{A} + \lambda I$ is an isomorphism of V onto V' and $A + \lambda I$ is just its (necessarily onto) restriction to $H \subset V'$. (More generally, if $\mathcal{R}$ is the Riesz map of H onto H' , then $A = \mathcal{R}^{-1}\mathcal{A}$.) Finally, note that A is accretive on H exactly when the linear operator $\mathcal{A}$ satisfies

$$\mathcal{A}v(v) \geq 0, \quad v \in V.$$

This property of $\mathcal{A}$ is called *monotone* and will predominate in Chapters II and III. Not every m -accretive A corresponds to a monotone $\mathcal{A}$ as above; those which do are a special class.

DEFINITIONS. Let V, H be Hilbert spaces with $H \cong H'$ and let $\mathcal{A} \in \mathcal{L}(V, V')$ be *monotone*:

$$\mathcal{A}v(v) \geq 0, \quad v \in V.$$

The corresponding unbounded operator on H , $A = \mathcal{A}|_{V \times H}$, is then accretive and we shall call it *regular accretive* when it is so determined by a triple $\{\mathcal{A}, V, H\}$. Assume further that for every $\varepsilon > 0$, $\mathcal{A} + \varepsilon I$ is V -elliptic. (This implies that $\mathcal{A}$ is monotone.) Then $\text{Rg}(\mathcal{A} + \varepsilon I) = V'$ for each $\varepsilon > 0$ and so $\text{Rg}(A + \varepsilon I) = H$, hence, A is m -accretive, and we shall call it *regular m -accretive*.

One can show that every accretive self-adjoint operator on H is regular m -accretive. For this, let V be the closure of $D(A)$ with the scalar product $(u, v)_V \equiv (u, v)_H + (Au, v)_H$.

I.5. The Cauchy Problem

Let H be a Hilbert space, D a subspace, and $A : D \rightarrow H$ an unbounded linear operator. The *Cauchy Problem* for the evolution equation

$$(5.1) \quad u'(t) + Au(t) = 0, \quad t > 0,$$

is to find a *solution* $u \in C([0, \infty), H) \cap C^1((0, \infty), H)$ such that $u(t) \in D(A)$ for $t > 0$ and $u(0) = u_0$, where $u_0 \in H$ is prescribed. The continuity or differentiability of the vector-valued function $u : [0, \infty) \rightarrow H$ is defined exactly as in the real-valued case $H = \mathbb{R}$, but with absolute-value replaced by the H norm.

Suppose that for each $u_0 \in D$ there is a unique solution u of the Cauchy Problem; then define $S(t)u_0 = u(t)$ for $t \geq 0$, $u_0 \in D$. Since A is linear it follows each $S(t) : D \rightarrow D$ is linear for $t \geq 0$. Furthermore, since the translate $u(t + \tau)$ is a solution of (5.1) for each $\tau \geq 0$, we find from the uniqueness that

$$S(t + \tau)u_0 = S(t)S(\tau)u_0, \quad t, \tau \geq 0; S(0) = I.$$

If $u(t) = S(t)u_0$ then

$$1/2 \frac{d}{dt} \|u(t)\|^2 = -(Au(t), u(t))_H,$$

so if A is accretive then $\|u(t)\|$ is decreasing for $t \geq 0$, hence, $\|S(t)u_0\| \leq \|u_0\|$ and thus each $S(t)$ is a contraction on D . If D is dense in H each $S(t)$ has a unique extension to a contraction on H and we obtain the following.

DEFINITION. $\{S(t) : t \geq 0\}$ is a *linear contraction semigroup* (or LCS) if $S(t) : H \rightarrow H$ is a linear contraction for each $t \geq 0$,

$$S(t + \tau) = S(t)S(\tau) \quad \text{for } t, \tau \geq 0, \quad S(0) = I,$$

$$\text{and} \quad S(\cdot)x \in C([0, \infty), H) \quad \text{for each } x \in H.$$

For example, suppose $A \in \mathcal{L}(H)$ and A is accretive. Then one can define by power series in $\mathcal{L}(H)$ the operator

$$\exp(A) = \sum_{n=0}^{\infty} A^n / n!$$

and thereby the family of operators $S(t) = \exp(-tA)$, $t \geq 0$. It follows that

$$\frac{d}{dt}S(t) = -A \cdot S(t) \quad \text{in } \mathcal{L}(H)$$

and hence $u(t) \equiv S(t)u_0$ is the solution of the Cauchy Problem for (5.1) with $u(0) = u_0$. Note that the operator $-A$ can be recovered from the semigroup by computing the right-derivative at $t = 0$,

$$-Au_0 = \lim_{t \rightarrow 0^+} \{(u(t) - u(0))/t\} \equiv D^+u(0) .$$

DEFINITION. The *generator* of the LCS $\{S(t) : t \geq 0\}$ is the operator B defined by $Bx \equiv D^+(S(0)x)$ for each x belonging to

$$D(B) = \{x \in H : \lim_{h \rightarrow 0^+} h^{-1}(S(h)x - x) \equiv D^+(S(0)x) \text{ exists}\} .$$

Our preceding remarks verify most of the following.

PROPOSITION 5.1. *Let $A \in L(D, H)$ be closed and accretive, D dense in H , and assume for every $u_0 \in D$ there exists a solution $u \in C^1([0, \infty), H)$ of (5.1) on $t \geq 0$ with $u(0) = u_0$. Construct $\{S(t) : t \geq 0\}$ as above, so $u(t) = S(t)u_0$, $t \geq 0$. Then $\{S(t) : t \geq 0\}$ is a LCS on H whose generator is an extension of $-A$.*

PROOF. Since A is accretive, each Cauchy problem has at most one solution, so the construction of $\{S(t) : t \geq 0\}$ is done as above. For each $u_0 \in D$ we have

$$S(t)u_0 - u_0 = \int_0^t u' = - \int_0^t Au(s) ds , \quad t > 0 ,$$

and the integrand is continuous on $[0, \infty)$, so $D^+(S(0)u_0) = -Au_0$. □

Our objective is to find sufficient conditions on an operator A in order that the Cauchy problem for (5.1) will have a solution. Thus we shall characterize those operators which are generators of linear contraction semigroups. To begin, suppose that $B : D(B) \rightarrow H$ is the generator of $\{S(t) : t \geq 0\}$ as above. Then for each $x \in D(B)$

$$S(t+h)x - S(t)x = (S(h) - I)S(t)x = S(t)(S(h) - I)x , \quad h > 0 , t \geq 0 ,$$

so dividing by h and letting $h \rightarrow 0$ shows

$$D^+S(t)x = BS(t)x = S(t)Bx , \quad x \in D(B) , t \geq 0 .$$

In particular, $S(t) : D(B) \rightarrow D(B)$. Also, for $0 < h < t$

$$S(t)x - S(t-h)x = S(t-h)(S(h) - I)x$$

and the $S(t-h)$ are uniformly bounded so we may divide this by h and take the limit to obtain

$$D^-S(t)x = S(t)Bx , \quad x \in D(B) , t > 0 ,$$

where D^- denotes the left-derivative.

LEMMA 5.1. *For each $x \in D(B)$ the function $S(\cdot)x$ belongs to $C^1([0, \infty), H)$, and for each $t \geq 0$, $S(t)x \in D(B)$ and*

$$S(t)x - x = \int_0^t BS(s)x \, ds = \int_0^t S(s) Bx \, ds .$$

COROLLARY 5.1. *B is closed.*

PROOF. If $x_n \in D(B)$, $x_n \rightarrow x$ and $Bx_n \rightarrow y$ in H , then

$$h^{-1}(S(h) - I)x_n = h^{-1} \int_0^h S(s) Bx_n \, ds , \quad h > 0 , \, n \geq 1 ,$$

so we let $n \rightarrow \infty$ and then $h \rightarrow 0$ to obtain $D^+S(0)x = y$. □

COROLLARY 5.2. *If $u_0 \in D(B)$ and $u(t) \equiv S(t)u_0$, $t \geq 0$, then $u \in C^1([0, \infty), H)$ satisfies $u'(t) = Bu(t)$, $t \geq 0$, and $u(0) = u_0$.*

LEMMA 5.2. *$D(B)$ is dense in H , $\int_0^t S(s)x \, ds \in D(B)$ and*

$$S(t)x - x = B \int_0^t S(s)x \, ds , \quad t \geq 0 , \, x \in H .$$

PROOF. For each $t \geq 0$ we set $x_t = \int_0^t S(s)x \, ds$. Then if $h > 0$ we have

$$\begin{aligned} S(h)x_t - x_t &= \int_0^t (S(h+s) - S(s))x \, ds \\ &= \int_h^{t+h} S(s)x \, ds - \int_0^t S(s)x \, ds \pm \int_t^h S(s)x \, ds \\ &= \int_t^{t+h} S(s)x \, ds - \int_0^h S(s)x \, ds . \end{aligned}$$

Dividing by $h > 0$ and taking the limit shows $x_t \in D(B)$ and $Bx_t = S(t)x - x$ as desired. Finally, note that $x_t/t \in D(B)$ and $x_t/t \rightarrow x$ as $t \rightarrow 0$, so $D(B)$ is dense. □

Fix $\lambda > 0$ and note that $\{e^{-\lambda t}S(t) : t \geq 0\}$ is a linear contraction semigroup whose generator is $B - \lambda$. We have let λ denote the operator λI . From Lemma 5.1 and Lemma 5.2 we obtain, respectively,

$$\begin{aligned} e^{-\lambda t}S(t)x - x &= \int_0^t e^{-\lambda s}S(s)(B - \lambda)x \, ds , \quad x \in D(B) , \, t \geq 0 , \\ e^{-\lambda t}S(t)y - y &= (B - \lambda) \int_0^t e^{-\lambda s}S(s)y \, ds , \quad y \in H , \, t \geq 0 . \end{aligned}$$

Take the limit as $t \rightarrow \infty$ in these to get

$$\begin{aligned} x &= \int_0^\infty e^{-\lambda s}S(s)(\lambda - B)x \, ds , \quad x \in D(B) \\ y &= (\lambda - B) \int_0^\infty e^{-\lambda s}S(s)y \, ds , \quad y \in H . \end{aligned}$$

The first shows $\lambda - B$ is one-to-one and the second that it is onto H , so it follows that

$$\|(\lambda - B)^{-1}y\| \leq \int_0^\infty e^{-\lambda s} ds \|y\| = \lambda^{-1} \|y\| .$$

This completes the proof of the first half of our main result.

THEOREM 5.1 (HILLE-YOSIDA). *A necessary and sufficient condition for B to be the generator of a linear contraction semigroup is that $D(B)$ is dense and $\lambda(\lambda - B)^{-1}$ is a contraction on H for every $\lambda > 0$.*

PROOF (CONTINUED). The sufficiency will be established in three steps. First we approximate B by $B_\lambda \in \mathcal{L}(H)$, then construct the semigroup $\{\exp(tB_\lambda) : t \geq 0\}$ for $\lambda > 0$, and finally show the limit $\lim_{\lambda \rightarrow \infty} \exp(tB_\lambda)$ is the desired linear contraction semigroup.

LEMMA 5.3. *Define $B_\lambda = \lambda B(\lambda - B)^{-1}$, $\lambda > 0$. Then $B_\lambda = -\lambda + \lambda^2(\lambda - B)^{-1} \in \mathcal{L}(H)$ and for each $x \in D(B)$, $\|B_\lambda x\| \leq \|Bx\|$, $\lim_{\lambda \rightarrow \infty} B_\lambda x = Bx$.*

PROOF. For $x \in D(B)$

$$\begin{aligned} (B_\lambda + \lambda)(\lambda - B)x &= \lambda^2 x \quad \text{and } \lambda - B \text{ is onto } H, \text{ so} \\ B_\lambda &= -\lambda + \lambda^2(\lambda - B)^{-1} . \end{aligned}$$

From this follows $(\lambda - B)B_\lambda = \lambda B$ and $B_\lambda = \lambda(\lambda - B)^{-1}B$. Since $\lambda(\lambda - B)^{-1}$ is a contraction, the desired estimate follows. Also, for $x \in D(B)$, $\|\lambda(\lambda - B)^{-1}x - x\| = \frac{1}{\lambda}\|B_\lambda x\| \rightarrow 0$, so $\lambda(\lambda - B)^{-1}x \rightarrow x$ for each $x \in D(B)$, hence, for each $x \in H$. Thus $B_\lambda \rightarrow Bx$ for each $x \in D(B)$. $\square$

LEMMA 5.4. *Define $S_\lambda(t) = \exp(tB_\lambda) \in \mathcal{L}(H)$, $t \geq 0$, $\lambda > 0$. Then $\{S_\lambda(t) : t \geq 0\}$ is a linear contraction semigroup with generator B_λ for each $\lambda > 0$. For $x \in H$, $S_\lambda(t)x$ converges in H as $\lambda \rightarrow \infty$, uniformly on each interval $[0, T]$, $T > 0$.*

PROOF. From $S_\lambda(t) = \exp(-\lambda t) \exp(\lambda^2(\lambda - B)^{-1}t)$ we obtain

$$\|S_\lambda(t)\| \leq \exp(-\lambda t) \exp(\lambda t) = 1$$

for $t \geq 0, \lambda > 0$. The remaining semigroup properties are obtained from calculus of power series in $\mathcal{L}(H)$. Similarly we obtain

$$S_\lambda(t) - S_\mu(t) = \int_0^t \frac{d}{ds} (S_\mu(t-s)S_\lambda(s)) ds = \int_0^t S_\mu(t-s)S_\lambda(s)(B_\lambda - B_\mu) ds$$

by calculus in $\mathcal{L}(H)$, and thus the estimate

$$\|S_\lambda(t)x - S_\mu(t)x\| \leq t\|B_\lambda(x) - B_\mu(x)\| , \quad x \in D(B) .$$

With Lemma 5.3 this shows that $\{S_\lambda(t)x\}$ is uniformly Cauchy in H for $0 \leq t \leq T$. Each $S_\lambda(t)$ is a contraction so the limit exists for each $x \in H$, uniformly on bounded intervals. $\square$

To finish the proof of Theorem 5.1, we define $S(t)x = \lim_{\lambda \rightarrow \infty} S_\lambda(t)x$. It follows that $\{S(t) : t \geq 0\}$ is a linear contraction semigroup. For $x \in D(B)$, $S_\lambda(\cdot)B_\lambda x \rightarrow S(\cdot)Bx$ uniformly on bounded intervals, so from

$$S_\lambda(t)x - x = \int_0^t S_\lambda(s)B_\lambda x ds$$

we obtain in the limit

$$S(t)x - x = \int_0^t S(s)Bx ds, \quad x \in D(B), \quad t \geq 0.$$

This shows $D^+S(0)x = Bx$, $x \in D(B)$, so B is a (possible) restriction of the generator of $\{S(t) : t \geq 0\}$. But $I - B$ is onto H , so B equals the generator. $\square$

From Proposition 4.2 we obtain the following.

COROLLARY 5.3. *A necessary and sufficient condition for $-A : D(A) \rightarrow H$ to be the generator of a linear contraction semigroup is that A be m -accretive.*

We consider two fundamental examples.

EXAMPLE 5.A. Let $H = L^2(0, 1)$ and $A = \partial$ on $D(A) = \{u \in H^1(0, 1) : u(0) = cu(1)\}$ with $|c| \leq 1$. We showed in Section 4 that A is m -accretive, so by Theorem 5.1 we see the initial-boundary-value problem

$$(5.2.a) \quad \partial_t u(x, t) + \partial_x u(x, t) = 0, \quad 0 < x < 1, \quad t \geq 0,$$

$$(5.2.b) \quad u(0, t) = cu(1, t)$$

$$(5.2.c) \quad u(x, 0) = u_0(x)$$

has a unique solution for each $u_0 \in D(A)$. An explicit representation for this solution can be easily found. Since any solution of (5.2.a) is of the form $u(x, t) = F(x - t)$, it follows that

$$u(x, t) = u_0(x - t), \quad 0 \leq t \leq x \leq 1,$$

and then (5.2.b) implies

$$u(x, t) = cu_0(1 + x - t), \quad x \leq t \leq x + 1.$$

By an easy induction we obtain

$$u(x, t) = c^n u_0(n + x - t), \quad n - 1 + x \leq t \leq n + x, \quad n \geq 1.$$

This representation of the solution gives some additional information. First, the Cauchy problem can be solved only if $u_0 \in D(A)$, because $u(\cdot, t) \in D(A)$ implies $u(\cdot, t)$ is (absolutely) continuous and this is possible only if u_0 satisfies the boundary condition (5.2.b). Second, the solution satisfies $u(\cdot, t) \in H^1(0, 1)$ for every $t \geq 0$ but will *not* belong to $H^2(0, 1)$ unless $\partial u_0 \in D(A)$. That is, we do not in general have $u(\cdot, t) \in H^2(0, 1)$, no matter how smooth the initial function u_0 may be. Finally, the representation above defines a solution of (5.2) on $-\infty < t < \infty$ by allowing n to be *any* integer. Thus, the problem can be solved backwards in time as well as forward. This is related to the fact that $-A$ generates a *group* of operators.

EXAMPLE 5.B. For our second example we take $H = L^2(\mathbb{R})$, $V = H^1(\mathbb{R})$ and define $a(u, v) = (\partial u, \partial v)_H$ for $u, v \in V$. Corollary 4.1 shows that the operator

$$A = -\partial^2, \quad D(A) = \{u \in V : \partial^2 u \in L^2(\mathbb{R})\}$$

is m -accretive. Thus by Theorem 5.1 we obtain existence and uniqueness for the initial-value problem

$$(5.3.a) \quad \partial_t u(x, t) = \partial_x^2 u(x, t), \quad -\infty < x < \infty, \quad t > 0,$$

$$(5.3.b) \quad u(x, 0) = u_0(x),$$

with $u_0 \in D(A)$. We shall obtain a useful representation for the solution. First note the only bounded solutions of the form $\exp(\alpha x + \beta t)$ are $\exp(\pm i\mu x - \mu^2 t)$, $\mu \in \mathbb{R}$. Taking real parts and then integrating all these together lead to

$$\tilde{u}(x, t) = \int_0^\infty e^{-\mu^2 t} \cos(\mu x) d\mu = \frac{1}{\sqrt{t}} \int_0^\infty e^{-\lambda^2} \cos\left(\frac{x\lambda}{\sqrt{t}}\right) d\lambda$$

and this is a solution for $t > 0$. To evaluate this integral, set

$$K(s) = \int_0^\infty e^{-\lambda^2} \cos(s\lambda) d\lambda$$

and check that $K'(s) = -(s/2)K(s)$, hence, $K(s) = M \exp(-s^2/4)$. Finally, $K(0) = M = \sqrt{\pi}/2$, so we have shown

$$\tilde{u}(x, t) = (\pi/4t)^{1/2} \exp(-x^2/4t).$$

By a re-scaling suggested by the following result, we are led to the special function

$$K(x, t) = (4\pi t)^{-1/2} \exp(-x^2/4t), \quad t > 0.$$

LEMMA 5.5.

- (i) K is $C^\infty(\mathbb{R} \times \mathbb{R}_+)$ and each $\partial_x^m \partial_t^n K$ converges to zero exponentially as $|x| \rightarrow \infty$.
- (ii) $\partial_t K = \partial_x^2 K$.
- (iii) $K > 0$, $\int_{\mathbb{R}} K(x, t) dx = 1$, $t > 0$.
- (iv) For $\delta > 0$, $\lim_{t \rightarrow 0} K(x, t) = 0$ uniformly for $|x| \geq \delta$.
- (v) For $\delta > 0$, $\lim_{t \rightarrow 0} \int_{|x| \geq \delta} K(x, t) dx = 0$.

PROOF.

- (i) This follows from l'Hopital's rule.
- (ii) Differentiate and check.
- (iii) By a change of variable this integral is

$$(4\pi t)^{-1/2} \int_{\mathbb{R}} \exp(-u^2) (4t)^{1/2} du$$

so the result follows.

- (iv) For $|x| \geq \delta$ we have $K(x, t) \leq (4\pi t)^{-1/2} \exp(-\delta^2/4t)$.
- (v) $\int_{|x| \geq \delta} K(x, t) dx = \pi^{-1/2} \int_{|u| \geq \delta/(4t)} e^{-u^2} du \rightarrow 0$ as $t \rightarrow 0$. □

PROPOSITION 5.2. Let u_0 be bounded and continuous on $\mathbb{R}$. Define

$$u(x, t) = \begin{cases} \int_{\mathbb{R}} K(x - \xi, t) u_0(\xi) d\xi, & t > 0, \\ u_0(x), & t = 0. \end{cases}$$

Then u is bounded and continuous on $\{(x, t) : t \geq 0\}$, it is infinitely differentiable on $\{(x, t) : t > 0\}$, and it satisfies (5.3).

PROOF. The function u is well-defined and C^∞ by (i), a solution of the heat equation by (ii), and from (iii) we obtain $\sup_{x,t}(u) \leq \sup_x u_0$, so u is bounded. It remains only to check the continuity on $\mathbb{R} \times [0, \infty)$.

Let $x_0 \in \mathbb{R}$ and $\varepsilon > 0$; choose $A > |x_0|$. Since u_0 is uniformly continuous on $[-A, A]$ there is a $\delta > 0$ such that $(x_0 - 2\delta, x_0 + 2\delta) \subset [-A, A]$ and $x, y \in [-A, A]$, $|x - y| < 2\delta$ imply $|u_0(x) - u_0(y)| < \varepsilon/2$. By a change of variable and (iii) we obtain

$$\begin{aligned} u(x, t) - u_0(x_0) &= \int_{|s| \geq \delta} K(s, t)(u_0(x + s) - u_0(x_0)) ds \\ &\quad + \int_{|s| \leq \delta} K(s, t)(u_0(x + s) - u_0(x_0)) ds . \end{aligned}$$

If $|x - x_0| < \delta$ then for $|s| \leq \delta$ we have

$$x + s \in [-A, A] , \quad |x + s - x_0| < 2\delta$$

so $|u_0(x + s) - u_0(x_0)| < \varepsilon/2$. Thus the second term above is bounded by $\varepsilon/2$ by (iii). Thus we have

$$|u(x, t) - u_0(x_0)| \leq \int_{|s| \geq \delta} K(s, t) ds \cdot 2 \sup_x |u_0| + \varepsilon/2$$

and the result follows from (v). $\square$

Example 5.B illustrates the *regularizing effects* that occur with evolutions governed by regular m -accretive operators. Consider the case of such an operator A which arises from a triple $\{a(\cdot, \cdot), V, H\}$ for which $a(\cdot, \cdot)$ is symmetric. Thus A is *self-adjoint*: $A = A^*$. Let $\{S(t) : t \geq 0\}$ be the semigroup generated by $-A$, $u_0 \in D(A)$, and $u(t) = S(t)u_0$. Thus, (5.1) holds and $u(0) = u_0$. We seek estimates which imply “regularity” of the solution $u(t)$. First, from $(Au(t), u(t))_H = -\frac{1}{2} \frac{d}{dt} \|(u(t))\|_H^2$ we obtain

$$(5.4) \quad \int_0^T a(u(t), u(t)) dt = \frac{1}{2} (\|u_0\|_H^2 - \|u(T)\|_H^2) .$$

For each $h > 0$ and $t > \tau > 0$, $u(t+h) - u(t) = S(t-\tau)(u(\tau+h) - u(\tau))$, so $S(t-\tau)$ being a contraction shows $\|u'(t)\|_H \leq \|u'(\tau)\|_H$, hence, $\|u'(\cdot)\|_H$ is non-increasing. We have

$$t\|u'(t)\|_H^2 = -\frac{t}{2} \frac{d}{dt} a(u(t), u(t)) = -\frac{d}{dt} \left(\frac{t}{2} a(u(t), u(t)) \right) + \frac{1}{2} a(u(t), u(t))$$

since $a(\cdot, \cdot)$ is symmetric, and this yields

$$(5.5) \quad \int_0^T t\|u'(t)\|_H^2 dt + \frac{T}{2} a(u(T), u(T)) = \frac{1}{2} \int_0^T a(u(t), u(t)) dt .$$

Using the non-increase of $\|u'(\cdot)\|_H$ and (5.4), we obtain

$$\|u'(T)\|_H^2 \frac{T^2}{2} \leq \frac{1}{4} \|u_0\|_H^2 ,$$

and this leads to the following *parabolic regularizing property* of the LCS.

THEOREM 5.2. *If A is regular m -accretive and self-adjoint, the generated LCS satisfies the following:*

$$S(t) \text{ maps } H \text{ into } D(A) , \text{ and } \|tAS(t)\|_{\mathcal{L}(H)} \leq \frac{1}{\sqrt{2}} , \quad t > 0 .$$

PROOF. Let $w \in H$ and $w_n \in D(A)$ for $n \geq 1$ with $w_n \rightarrow w$. We have $S(t)w_n \rightarrow S(t)w$ and $\|AS(t)(w_m - w_n)\|_H \leq \|w_m - w_n\|_H / \sqrt{2}t$, so $\{AS(t)w_n\}$ converges in H . But A is closed, and so the desired result follows. $\square$

In the proof of Lemma 5.1 we found that A commutes with $S(t)$ on $D(A)$, $AS(t)x = S(t)Ax \in D(A)$ for $x \in D(A)$, so it follows that $S(t)$ maps $D(A)$ into $D(A^2)$ for $t > 0$. Thus $S(t) = S(t/2)S(t/2)$ maps H into $D(A^2)$ with

$$\|A^2S(t)\|_{\mathcal{L}(H)} = \|AS(t/2)AS(t/2)\|_{\mathcal{L}(H)} \leq \left(\frac{2}{t\sqrt{2}} \right)^2 .$$

By induction we obtain the following.

COROLLARY 5.4. *For every $t > 0$ and integer $p \geq 1$*

$$S(t) \text{ maps } H \text{ into } D(A^p) , \text{ and } \|A^pS(t)\|_{\mathcal{L}(H)} \leq \left(\frac{p}{t\sqrt{2}} \right)^p .$$

This gives the *spatial regularity* of the solution $u(t)$ of the Cauchy problem.

COROLLARY 5.5. *For every $u_0 \in H$ there is a unique solution $u \in C([0, \infty), H) \cap C^\infty((0, \infty), H)$ of (5.1) with $u(0) = u_0$, and it satisfies $u(t) \in D(A^p)$ for every $t > 0$ and $p \geq 1$.*

For any m -accretive operator, $A + I$ is a bijection of $D(A)$ onto H , and if we define a norm on $D(A)$ by

$$\|x\|_D \equiv (\|x\|_H^2 + \|Ax\|_H^2)^{1/2} , \quad x \in D(A) ,$$

it follows that $D(A)$ is a Hilbert space isomorphic to H . Similarly, $D(A^p)$ is a Hilbert space with scalar-product

$$(x, y)_{D^p} \equiv (x, y)_H + (A^p x, A^p y)_H , \quad x, y \in D(A^p) ,$$

and $(A + I)^p$ is an isomorphism of $D(A^p)$ onto H . For the special case of a self-adjoint regular accretive operator as above we can deduce from the identity

$$A^p \frac{u(t+h) - u(t)}{h} = \frac{S(t-\varepsilon+h) - S(t-\varepsilon)}{h} A^p S(\varepsilon) u_0 , \quad 0 < \varepsilon < t , h > 0 ,$$

that $\lim_{h \rightarrow 0+} \left(\frac{u(t+h) - u(t)}{h} \right) = u'(t)$ in the space $D(A^p)$. When A is a differential operator this shows $u'(t)$ agrees with the partial derivative with respect to time and a corresponding *temporal regularity* of the solution of (5.1).

I.6. Wave Equations

Consider again the vertical displacement within a fixed plane of a string of length $\ell > 0$. As before, let $T > 0$ be the tension applied to stretch the string. Here we shall study the time-dependent displacement $u(x, t)$ for $t > 0$, $0 < x < \ell$, and describe the vibrations of the string resulting from an initial displacement $u(x, 0) = u_0(x)$ and an initial velocity $u_t(x, 0) = u_1(x)$ as well as specific boundary conditions at the endpoints $x = 0, \ell$.

To derive the equation for these transverse vibrations of the string, recall from Section 3 that for a section of the string, $x_1 < x < x_2$, during the time interval, $t_1 < t < t_2$, the total impulse is given by

$$\int_{t_1}^{t_2} T(u_x(x_2, s) - u_x(x_1, s)) ds .$$

If the density of the string at x is given by $\rho > 0$, the change in momentum of this section is just

$$\int_{x_1}^{x_2} \rho(u_t(x, t_2) - u_t(x, t_1)) dx$$

and so this equals the impulse by Newton's second law. For a sufficiently smooth displacement $u(x, t)$, the equality of these integrals for each such $x_1 < x_2$ and $t_1 < t_2$ shows that the displacement satisfies the *wave equation*

$$(6.1) \quad \frac{\partial^2}{\partial t^2}(\rho u(x, t)) - T \frac{\partial^2}{\partial x^2} u(x, t) = 0, \quad 0 < x < \ell, \quad t > 0 .$$

Notice that the total energy of the string at each time t is given by the sum

$$\frac{1}{2} \int_0^\ell \rho u_t^2(x, t) dx + \frac{1}{2} \int_0^\ell T u_x^2(x, t) dx$$

of kinetic and potential energy which are determined, respectively, by the velocity u_t and vertical force $T u_x$. We shall show that various mixed initial-boundary-value problems for (6.1) are well-posed by reducing this second-order evolution equation to an equivalent first-order system to which Theorem 5.1 directly applies. This abstract development will be followed by a specific example for which we can display the solution explicitly and compare its properties with those guaranteed by the theory.

We shall regard (6.1) as a special case of the evolution equation

$$(6.2) \quad u''(t) + B u'(t) + A u(t) = 0, \quad 0 < t,$$

where A is an unbounded operator with domain D in a Hilbert space H and B is another operator. To obtain (6.1) from (6.2) we shall choose A to correspond to an appropriate Dirichlet or Neumann problem as in Corollary 4.1; thus, A will be regular m -accretive. The operator B will be continuous and linear from V to H and accretive on H . Such an operator arises naturally from models of friction in the wave equation and it also is convenient to include it in our discussion for the following reason. The function u is a solution of (6.2) if and only if the function w given by $w(t) = e^{-\lambda t} u(t)$, $t > 0$, satisfies

$$(6.3) \quad w''(t) + (2\lambda I + B)w'(t) + (\lambda^2 I + \lambda B + A)w(t) = 0, \quad t > 0 .$$

Thus, (6.2) and (6.3) are equivalent, and this observation permits us to relax the hypothesis on A and B . Specifically, it suffices to require that the operators $\lambda_0^2 I + \lambda_0 B + A$ and $2\lambda_0 I + B$ satisfy certain conditions for some λ_0 .

As in Section 4, let V be a Hilbert space which is dense and continuously imbedded in another Hilbert space H . Let $a(\cdot, \cdot)$ be a continuous, symmetric, bilinear and elliptic form on V , and let $A : D \rightarrow H$ be the unbounded operator on H constructed in Proposition 4.1; that is, A is the regular m -accretive operator constructed from $a(\cdot, \cdot)$, V and H with

$$(6.4) \quad a(u, v) = (Au, v)_H, \quad u \in D, \quad v \in V.$$

Note that $a(\cdot, \cdot)$ is a scalar-product on V whose norm is equivalent to the V -norm; hereafter we take $a(\cdot, \cdot)$ as the scalar-product on V . Let $B : V \rightarrow H$ be continuous and linear, and suppose B is accretive in H :

$$(Bu, u)_H \geq 0, \quad u \in V.$$

Consider the product space $\mathcal{H} = V \times H$ with the scalar-product

$$(\vec{u}, \vec{v})_{\mathcal{H}} = a(u_1, v_1) + (u_2, v_2)_H, \quad \vec{u}, \vec{v} \in \mathcal{H}.$$

This is a Hilbert space, and we define on it the operator $\mathbb{A} : \mathcal{D} \rightarrow \mathcal{H}$ given by

$$\mathbb{A}\vec{u} = \begin{pmatrix} -u_2 \\ Au_1 + Bu_2 \end{pmatrix}, \quad \vec{u} \in \mathcal{D},$$

with domain $\mathcal{D} = D \times V$. This is the operator that arises when we write (6.2) as a first-order system

$$\begin{aligned} u'(t) - v(t) &= 0 \\ v'(t) + Au(t) + Bv(t) &= 0, \end{aligned}$$

for then the function $\vec{u}(t) = [u(t), v(t)]$, $t > 0$, is a solution of the equation

$$(6.5) \quad \vec{u}'(t) + \mathbb{A}\vec{u}(t) = \vec{0}, \quad t > 0,$$

in the space $\mathcal{H}$. Conversely, the first component of a solution of (6.5) satisfies the second-order (6.2).

Our plan is to show the Cauchy problem for (6.5) is well-posed; by Theorem 5.1 it is sufficient to show that $\mathbb{A}$ is m -accretive on $\mathcal{H}$. First note that for $\vec{u} \in \mathcal{D}$

$$\begin{aligned} (\mathbb{A}\vec{u}, \vec{u})_{\mathcal{H}} &= -a(u_2, u_1) + (Au_1 + Bu_2, u_2)_H \\ &= (Bu_2, u_2)_H, \end{aligned}$$

so $\mathbb{A}$ is accretive. Note the critical dependence of this calculation on the choice of the scalar-product on V and $\mathcal{H}$, and that $\mathbb{A}$ is accretive exactly when B is accretive. To show that the range of $\lambda I + \mathbb{A}$ is $\mathcal{H}$, we let $\vec{f} = [f_1, f_2] \in \mathcal{H}$ be given and seek a $\vec{u} = [u_1, u_2] \in \mathcal{D}$ such that $\lambda \vec{u} + \mathbb{A}\vec{u} = \vec{f}$, i.e.,

$$u_1 \in D, \quad u_2 \in V : \lambda u_1 - u_2 = f_1, \quad \lambda u_2 + Au_1 + Bu_2 = f_2.$$

This system is equivalent to

$$\begin{aligned} u_1 \in D : (\lambda^2 I + \lambda B + A)u_1 &= (\lambda + B)f_1 + f_2, \\ u_2 &= \lambda u_1 - f_1, \end{aligned}$$

so it suffices to show that the range of $\lambda^2 I + \lambda B + A$ is all of H . But the bilinear form $\tilde{a}(\cdot, \cdot)$ defined on V by

$$\tilde{a}(u, v) = \lambda^2(u, v)_H + \lambda(Bu, v)_H + a(u, v) , \quad u, v \in V$$

is continuous and coercive, so the desired result follows from Theorem 2.2. That is, for each $h \in H$ there is a unique

$$u \in V : \tilde{a}(u, v) = (h, v)_H , \quad v \in V ,$$

and this is equivalent to

$$u \in D : (\lambda^2 I + B + A)u = h .$$

Thus, $\mathbb{A}$ is m -accretive on $\mathcal{H}$ and this leads to the following result.

PROPOSITION 6.1. *Let V and H be Hilbert spaces with V dense and continuously imbedded in H . Let $a(\cdot, \cdot)$ be a continuous, bilinear and symmetric form on V and let $B \in \mathcal{L}(V, H)$ for which either B is symmetric and there are numbers $\lambda_0, c > 0$ such that*

$$\begin{aligned} a(u, u) + \lambda_0(Bv, v)_H + \lambda_0^2 \|v\|_H^2 &\geq c \|v\|_V^2 , & v \in V , \\ (Bv, v)_H + 2\lambda_0 \|v\|_H^2 &\geq 0 , & v \in V , \end{aligned}$$

or else the above hold with $\lambda_0 = 0$. Let $A : D \rightarrow H$ be the operator constructed from $a(\cdot, \cdot)$, V and H according to (6.4). Then for each pair $u_0 \in D$, $u_1 \in V$ there is a unique solution $u \in C^1([0, \infty), V) \cap C^2([0, \infty), H)$ of

$$(6.6.a) \quad u''(t) + Bu'(t) + Au(t) = 0 , \quad t \geq 0$$

$$(6.6.b) \quad u(0) = u_0 , \quad u'(0) = u_1 .$$

PROOF (CONTINUED). By our remarks leading to (6.3), if B is symmetric it suffices to let $\lambda_0 = 0$ in our hypotheses. Since $\mathcal{A}$ is m -accretive the Cauchy problem for (6.5) with $\vec{u}(0) = [u_0, u_1]$ has a unique solution $\vec{u} \in C^1([0, \infty), \mathcal{H})$. Then the first component is the corresponding solution of (6.6). $\square$

Example. We shall apply Proposition 6.1 to resolve the initial-boundary-value problem for the wave equation with null displacement at the endpoints. After rescaling the coefficients and interval, this problem has the form

$$(6.7.a) \quad \partial_t^2 u(x, t) - \partial_x^2 u(x, t) = 0 , \quad 0 < x < 1 , t > 0 ,$$

$$(6.7.b) \quad u(0, t) = u(1, t) , \quad t > 0 ,$$

$$(6.7.c) \quad u(x, 0) = u_0(x) , \quad \partial_t u(x, 0) = u_1(x) , \quad 0 < x < 1 .$$

Set $H = L^2(0, 1)$, $V = H_0^1(0, 1)$, and define $a(u, u) = \int_0^1 \partial u(x) \partial v(x) dx$ on V , so the corresponding regular m -accretive operator is given by $A = -\partial^2$ on $D(A) = H_0^1(0, 1) \cap H^2(0, 1)$. According to Proposition 6.1, there is a unique solution $u \in C^1([0, \infty), H_0^1(0, 1)) \cap C^2([0, \infty), L^2(0, 1))$ if the initial data satisfies $u_0 \in D(A)$, $u_1 \in V$.

For the special case of (6.7) we can find an explicit representation for the solution. Extend the functions u_0 and u_1 from the interval $(0, 1)$ to all of $\mathbb{R}$ so they are *odd* with respect to both of the endpoints, $x = 0$, $x = 1$. These extensions are 2-periodic, and the above requirements on the initial data in Proposition 6.1 are

sufficient to make the extensions of u_0 , ∂u_0 and u_1 all continuous on $\mathbb{R}$. Using these extensions, we define

$$u(x, t) = \frac{1}{2}(u_0(x+t) + u_0(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} u_1(s) ds .$$

Due to the indicated continuity conditions, among others, it follows that u is the solution to (6.7) given by Proposition 6.1. This representation shows that the conditions imposed on the initial data are appropriate. Since the problem is *reversible*, that is, it is well-posed for all $t \in \mathbb{R}$, it follows that these conditions are actually *necessary* for a solution as given above.

CHAPTER II

Nonlinear Stationary Problems

II.1. Banach Spaces

Banach space is a complete normed linear space X . Its *dual space* X' is the linear space of all continuous linear functionals $f : X \rightarrow \mathbb{R}$, and it has norm $\|f\|_{X'} \equiv \sup\{|f(x)| : \|x\| \leq 1\}$; X' is also a Banach space. We shall denote the value of $f \in X'$ at $x \in X$ by either $f(x)$ or $\langle f, x \rangle$. Likewise from X' we construct the *bidual* or *second dual* $X'' = (X')'$. Furthermore, with each $x \in X$ we can define $\varphi(x) \in X''$ by $\varphi(x)(f) \equiv f(x)$, $f \in X'$; this satisfies clearly $\|\varphi(x)\| \leq \|x\|$. Moreover, for each $x \in X$ there is an $f \in X'$ with $f(x) = \|x\|$ and $\|f\| = 1$, so it follows that $\|\varphi(x)\| = \|x\|$. Since φ is linear we see that $\varphi : X \rightarrow X''$ is a linear isometry of X onto a closed subspace of X'' ; we denote this by $X \hookrightarrow X''$. If φ is onto X'' we say X is *reflexive*: $X \cong X''$. Closed subspaces, duals and products of reflexive spaces are likewise reflexive.

Convergence in X is the usual *norm convergence* or *strong convergence*: a sequence $\{x_n\}$ in X converges to x if $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$, and this is denoted by $x_n \rightarrow x$ or $\lim_{n \rightarrow \infty} x_n = x$. This is related to the strong topology on X with neighborhood basis $B(r) \equiv \{x \in X : \|x\| < r\}$, $r > 0$, at the origin. There is also a *weak topology* on X obtained from the base of neighborhoods $B(r; f_1, f_2, \dots, f_n) \equiv \{x \in X : |f_j(x)| < r, 1 \leq j \leq n\}$, $r > 0$, $f_j \in X'$. This is the weakest topology on X for which every $f \in X'$ is continuous; a (net or) sequence $\{x_n\}$ in X is *weakly convergent* to x if $\lim_{n \rightarrow \infty} f(x_n) = f(x)$ for every $f \in X'$, and this is denoted by $x_n \rightharpoonup x$ or $w - \lim_{n \rightarrow \infty} x_n = x$. Finally, if X happens to be a dual space, say $X = Y'$, there is also the *weak* topology* on X obtained from the neighborhood basis $B(r; f_1, \dots, f_n)$ as above for $r > 0$ but $f_j = \varphi(y_j)$, $1 \leq j \leq n$, i.e., $\tilde{B}(r; y_1, \dots, y_n) = \{x \in X : |x(y_j)| < r, 1 \leq j \leq n\}$. This is precisely pointwise convergence, and it is weaker than the weak topology on X ; it is equivalent to it exactly when X is reflexive.

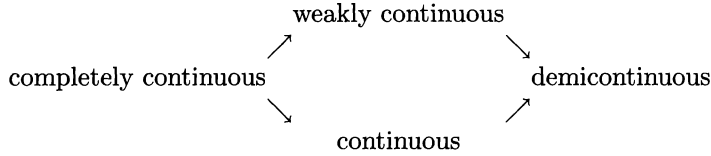
Finally, we remark on compactness in the three topologies. First, the unit ball $B \equiv \{x \in X : \|x\| \leq 1\}$ in X is compact if and only if $\dim(X) < \infty$. (The same holds for B' in X' , of course.) Second, B' is weakly compact in X' if and only if X is reflexive, and, third, B' is always weakly* compact in X' . (This follows since B' is a closed subspace of a product of compact spaces.) The following very deep result is important in many applications; see [59, p.141].

THEOREM 1.1 (EBERLEIN-SHMULYAN). *A Banach space is reflexive if and only if it is sequentially weakly compact, i.e., every bounded sequence contains a weakly convergent subsequence.*

A set $S \subset X$ is *bounded* if $S \subset B(r)$ for some $r > 0 : x \in S \Rightarrow \|x\| < r$. Likewise it is *weakly bounded* if some multiple of each weak neighborhood base contains S : for each $f \in X'$ there is an r for which $x \in S \Rightarrow |f(x)| < r$.

THEOREM 1.2 (UNIFORM BOUNDEDNESS). *A set S in Banach space is weakly bounded if and only if it is bounded.*

Let X and Y be Banach spaces and $T : X \rightarrow Y$ a function. Then T is *continuous* (weakly continuous) if it is continuous with X and Y each endowed with their corresponding norm (resp., weak) topologies. Similar terminology is used for weak* when both are dual spaces, and the modifier “sequentially” means that sequences (not nets) are considered. T is called *demicontinuous* if it is continuous from X with norm convergence to Y with weak convergence, and it is called *completely continuous* if it is continuous from X with weak to Y with strong convergence.



The function T is *bounded* if S bounded in X implies the image $T(S)$ is bounded in Y , and it is *compact* if it is continuous and S bounded implies $T(S)$ is *precompact*, i.e., its closure is compact.

Assume hereafter that X is reflexive and $T : X \rightarrow Y$.

LEMMA 1.1. *If T is completely continuous, then it is compact.*

PROOF. If S is bounded in X then a sequence in $T(S)$ is given by $\{T(x_n)\}$ with $\{x_n\}$ in S ; a subsequence $x_{n'} \rightarrow x$ in X so $T(x_{n'}) \rightarrow T(x)$ in Y . $\square$

Conversely, let's suppose $T : X \rightarrow Y$ is compact. If $x_n \rightarrow x$ in X then $\{x_n\}$ is bounded, weakly and strongly. If it is *not* that $T(x_n) \rightarrow T(x)$, there is a subsequence $\{x_{n'}\}$ for which $T(x_{n'})$ remains outside a strong neighborhood of $T(x)$. Now $\{x_{n'}\}$ bounded implies $\{T(x_{n'})\}$ contains a convergent subsequence, say $T(x_{n''}) \rightarrow y$, and this is a contradiction if $y = T(x)$. Thus we have proved the

LEMMA 1.2. *If T is compact and weakly sequentially continuous, then T is completely sequentially continuous.*

Note that T being weakly sequentially continuous implies that T is bounded. Also we needed above only that the graph of T be *closed* in $X_w \times Y_s$. Here X_w denotes X with weak convergence, and Y_s is Y with strong convergence.

Finally, note that complete continuity or compactness is so severe a restriction that it never holds for the identity in an infinite-dimensional Hilbert space. That is, if $\{e_n\}$ is an orthonormal basis for Hilbert space X , then for each $x \in X$ we have $x = \sum_{n=1}^{\infty} (e_n, x) e_n$ and $\|x\|^2 = \sum_{n=1}^{\infty} |(e_n, x)|^2$, so $\lim_{n \rightarrow \infty} (e_n, x) = 0$ for each $x \in X \cong X'$. Thus $e_n \rightarrow 0$, but $\|e_n\| = 1$ so it is impossible for any subsequence to be strongly convergent.

Assume hereafter that $T : X \rightarrow Y$ is *linear*. If T is bounded then by linearity there is a $K > 0$: $\|T(x)\| \leq K\|x\|$ for all $x \in X$ and (again, by linearity) this is equivalent to (uniform) continuity. This proves that T is bounded if and only if it

is continuous. Recall the *dual* or *adjoint* of T is the linear $T' : Y' \rightarrow X'$ defined by $T'(f)(x) = f(T(x))$ for $f \in Y'$, $x \in X$. Thus, $T'(f) = f \circ T \in X'$. If the net $x_\alpha \rightarrow x$ in X and $f \in Y'$, then $f(T(x_\alpha)) \equiv (T'f)(x_\alpha) \rightarrow (T'f)(x) \equiv f(T(x))$, so $Tx_\alpha \rightarrow Tx$ in Y . This shows that T being continuous implies that T is weakly continuous. Suppose X is reflexive. If T is not bounded, there is a bounded sequence $\{x_n\}$ such that $\|Tx_n\| \rightarrow \infty$. But then there is a weakly convergent subsequence $\{x_{n'}\}$ for which $\{T(x_{n'})\}$ is not bounded and, hence, not weakly convergent. This shows that T being weakly continuous implies that T is bounded. Thus, for a linear operator in reflexive Banach space, we have the equivalence of continuity, weak continuity and boundedness.

II.2. Existence Theorems

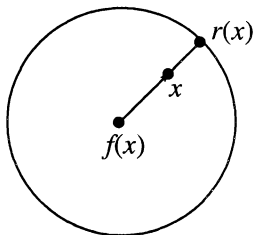
We begin with a fundamental fixed-point result; see [44, p. 82].

THEOREM (BROUWER). *If f is a continuous map of the closed unit ball B^n of $\mathbb{R}^n$ into itself, then there exists an $x \in B^n$: $f(x) = x$.*

There is another form of this result which is more geometric.

THEOREM. *There does not exist a continuous function r of B^n into its boundary ∂B^n with $r(x) = x \forall x \in \partial B^n$.*

Such a function r is called a *retract* of B^n . These two results are equivalent. To see this, let r be a retract of B^n . Define the antipodal map $a : \partial B^n \rightarrow \partial B^n$ by $a(x) = -x$. Then $a \circ r : B^n \rightarrow B^n$ has no fixed point, so the Brouwer Theorem implies the No-Retract Theorem. Conversely, suppose $f : B^n \rightarrow B^n$ has no fixed point: $f(x) \neq x$ for $x \in B^n$. Then for each $x \in B^n$, there is a unique $r(x) \in \partial B^n$: $r(x) = x + t(x - f(x))$ with $t \geq 0$, and r is a retract of B^n .



A special and useful case is the following intermediate-value theorem:

PROPOSITION 2.1. *Let $f : B^n \rightarrow \mathbb{R}^n$ be continuous and $(f(x), x) \geq 0$ for every $x \in \partial B^n$. Then f has a zero.*

PROOF. Otherwise, the map $x \mapsto -f(x)/\|f(x)\|$ of B^n into B^n has a fixed point $x_0 = -f(x_0)/\|f(x_0)\|$ for which $(f(x_0), x_0) = -\|f(x_0)\| < 0$. $\square$

This is an example of how an *a-priori* estimate and continuity lead to the existence of a solution to a nonlinear equation, $f(x) = 0$. We shall obtain such results in the setting of Banach space where $f : K \rightarrow K$ is continuous and K is a compact convex set.

Let V be a reflexive Banach space; consider a function $\mathcal{A} : V \rightarrow V'$. We say that $\mathcal{A}$ is *monotone* if $\langle \mathcal{A}(u) - \mathcal{A}(v), u - v \rangle \geq 0 \forall u, v \in V$, and *hemicontinuous* if for each $u, v \in V$ the real-valued function $t \mapsto \mathcal{A}(u + tv)(v)$ is continuous. (Clearly

this last condition is true if the restriction of $\mathcal{A}$ to each line segment is continuous into V' with weak convergence.)

DEFINITION. $\mathcal{A}$ is *type M* if $u_n \rightharpoonup u$, $\mathcal{A}u_n \rightharpoonup f$ and $\limsup \mathcal{A}u_n(u_n) \leq f(u)$ imply that $\mathcal{A}u = f$.

LEMMA 2.1. *If $\mathcal{A}$ is hemicontinuous and monotone then it is type M.*

PROOF. Let $\{u_n\}$ be given as above. By monotonicity, $\langle \mathcal{A}u_n - \mathcal{A}v, u_n - v \rangle \geq 0$, $v \in V$ so we obtain $\langle f - \mathcal{A}v, u - v \rangle \geq 0$ for all $v \in V$. For any $w \in V$ we set $v = u - tw$ with $t > 0$ and let t converge to zero; hemicontinuity implies $(f - \mathcal{A}u)(w) \geq 0$ for all $w \in V$, so $\mathcal{A}u = f$. $\square$

LEMMA 2.2. *If $\mathcal{A}$ is type M and bounded then it is demicontinuous.*

PROOF. Let $u_n \rightharpoonup u$ in V . Since $\mathcal{A}$ is bounded there is a subsequence $\{u_{n'}\}$ for which $\mathcal{A}(u_{n'}) \rightharpoonup f$. But $|\mathcal{A}u_{n'}(u_{n'}) - f(u)| \leq |(\mathcal{A}u_{n'} - f)(u)| + \|\mathcal{A}u_{n'}\| \|u_{n'} - u\| \rightarrow 0$ so $\lim_{n \rightarrow \infty} \mathcal{A}u_{n'}(u_{n'}) = f(u)$. Since $\mathcal{A}$ is type M it follows that $\mathcal{A}u = f$; since a subsequence as above can be extracted from *any* subsequence of $\{u_n\}$, it follows that $\mathcal{A}u_n \rightharpoonup \mathcal{A}(u)$. $\square$

COROLLARY 2.1. *Let V_0 be a finite dimensional space, $j : V_0 \rightarrow V$ an injection and $j' : V' \rightarrow V'_0$ the dual operator. Then $j' \mathcal{A} j : V_0 \rightarrow V'_0$ is continuous.*

THEOREM 2.1. *Let V be separable reflexive Banach space and $f \in V'$. Assume $\mathcal{A} : V \rightarrow V'$ is type M, bounded, and there is a $\rho > 0$ such that*

$$(2.1) \quad \mathcal{A}v(v) > f(v) \quad \forall v \in V : \|v\| > \rho .$$

Then $f \in Rg(\mathcal{A})$: there exists $u \in V$: $\mathcal{A}u = f$.

PROOF. We use Galerkin's method to reduce the problem to the finite-dimensional case. Let $\{w_1, w_2, \dots\}$ be an independent set of vectors whose linear span is dense in V , and let V_m denote the linear span of $\{w_1, w_2, \dots, w_m\}$ for each $m \geq 1$. Let $j_m : \mathbb{R}^m \rightarrow V_m$ denote representation with respect to this basis.

Fix $m \geq 1$ and consider the approximate problem in V'_m ,

$$u_m \in V_m : \mathcal{A}(u_m)(w_j) = f(w_j) , \quad 1 \leq j \leq m .$$

This is equivalent to finding $u_m = j_m(\hat{u}_m)$, where

$$\hat{u}_m \in \mathbb{R}^m : F(\hat{u}_m) \equiv (j'_m \mathcal{A} j_m)(\hat{u}_m) - j'_m f = 0 .$$

But $F : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is continuous and satisfies $(F(u), u)_{\mathbb{R}^m} \geq 0$ for $\|u\| \geq \rho$ so F has a zero.

For each $m \geq 1$ we have a u_m as above and, hence, $\mathcal{A}u_m(u_m) = f(u_m)$. Again from (2.1) we find $\|u_m\| \leq \rho$, and $\mathcal{A}$ is bounded so there is a subsequence (denoted by $\{u_m\}$ again) for which $u_m \rightharpoonup u$, $\mathcal{A}(u_m) \rightharpoonup f$ and $\mathcal{A}u_m(u_m) = f(u_m) \rightarrow f(u)$. Since $\mathcal{A}$ is type M, $\mathcal{A}(u) = f$. $\square$

DEFINITION. The function $\mathcal{A} : V \rightarrow V'$ is *coercive* if

$$\frac{\mathcal{A}u(u)}{\|u\|} \rightarrow \infty \quad \text{as } \|u\| \rightarrow \infty .$$

COROLLARY 2.2. *If $\mathcal{A}$ is type M , bounded and coercive on a separable reflexive Banach space to its dual then $\mathcal{A}$ is surjective.*

EXAMPLE. Let $A : \mathbb{R} \rightarrow \mathbb{R}$ be monotone, continuous and $f \in \mathbb{R}$. Then “ $A(r)r > fr$ for large $|r|$ ” if and only if f belongs to the interior of $Rg(A)$.

REMARK. An operator $\mathcal{A}$ is called *locally bounded* if the image of any convergent sequence is bounded. The proof of Lemma 2.2 shows that this is sufficient with type M to obtain demicontinuity. Thus, for monotone $\mathcal{A}$, demicontinuity is equivalent to hemicontinuity and local boundedness. A deeper result of Browder and Rockafellar is that monotonicity implies local boundedness, so for monotone operators, hemicontinuity and demicontinuity are equivalent.

If uniqueness holds for the equation $\mathcal{A}(u) = f$, then the *original sequence* $\{u_m\}$ in the proof of Theorem 2.1 must converge (weakly) to u . This follows from an argument similar to the proof of Lemma 1.2. Sufficient conditions for uniqueness are the following.

DEFINITIONS. $\mathcal{A}$ is *strictly monotone* if $\langle \mathcal{A}u - \mathcal{A}v, u - v \rangle > 0$ for all $u \neq v$ in V and $\mathcal{A}$ is *strongly monotone* if there is a $c > 0$ for which

$$\langle \mathcal{A}u - \mathcal{A}v, u - v \rangle \geq c\|u - v\|_V^2, \quad u, v \in V.$$

Note that strongly monotone implies coercive and that in Theorem 2.1 we have $u_m \rightarrow u$ (strongly). Moreover, for monotone operators we have the following useful characterization of solutions.

PROPOSITION 2.2 (MINTY). *If $\mathcal{A} : V \rightarrow V'$ is monotone and hemicontinuous, then $\mathcal{A}u = f$ if and only if $\langle \mathcal{A}v - f, v - u \rangle \geq 0$ for all $v \in V$. If also $\mathcal{A}$ is bounded and satisfies (2.1), then the set of solutions $U \equiv \{u \in V : \mathcal{A}u = f\}$ is closed, convex, non-empty and bounded.*

PROOF. Clearly $\mathcal{A}u = f$ implies $\langle \mathcal{A}v - f, v - u \rangle \geq 0$ for $v \in V$. Conversely, set $v = u + tw$ and let t converge to zero to obtain $\langle \mathcal{A}u - f, w \rangle \geq 0$ for all $w \in V$. Thus $U = \bigcap_{v \in V} C_v$ where $C_v = \{u \in V : \langle \mathcal{A}v - f, v - u \rangle \geq 0\}$ is closed and convex. If $\mathcal{A}$ satisfies (2.1) then U is bounded and, by Theorem 2.1, $\mathcal{A}$ also bounded implies U is non-empty. $\square$

DEFINITION. $\mathcal{A}$ is *maximal monotone* if it is monotone and $\mathcal{A}u = f$ if (and only if) $\langle \mathcal{A}v - f, v - u \rangle \geq 0$ for all $v \in V$.

Thus $\mathcal{A}$ has no monotone proper extension. From Proposition 2.2 and the proof of Lemma 2.1 it follows that monotone and hemicontinuous imply maximal monotone which in turn implies type M .

A useful variation on Theorem 2.1 is given next; although boundedness of $\mathcal{A}$ is relaxed, $\mathcal{A}$ is locally bounded by the preceding Remark.

THEOREM 2.2 (MINTY-BROWDER). *Let V be a separable reflexive Banach space and $f \in V'$. Assume $\mathcal{A} : V \rightarrow V'$ is monotone, demicontinuous and that (2.1) holds. Then $f \in Rg(\mathcal{A})$.*

PROOF. As before we obtain a sequence $u_m \in V_m$ such that $\langle \mathcal{A}u_m - f, w \rangle = 0$ for $w \in V_m$ and $u_m \rightharpoonup u$. If $v \in V_n$ and $m \geq n$, then $0 \leq \langle \mathcal{A}v - \mathcal{A}u_m, v - u_m \rangle = \langle \mathcal{A}v - f, v - u_m \rangle$ since $v - u_m \in V_m$, hence, $0 \leq \langle \mathcal{A}v - f, v - u \rangle$ for all $v \in \bigcup_{n \geq 1} V_n$.

The same holds for all $v \in V$ by demicontinuity since $\bigcup_{n \geq 1} V_n$ is dense in V , and the maximal monotonicity shows $\mathcal{A}u = f$. $\square$

We have shown that a monotone continuous operator is type M and have proven an existence result for *equations* with these rather general operators. Next we give some examples of type M operators, one of which indicates that this class is very sensitive to perturbations, and then we introduce the more stable class of *pseudo-monotone operators* for which we can resolve a wider class of problems, namely, *inequalities*. In Section 6 we shall present a variety of examples of variational inequalities and then introduce a class of elliptic operators which are *quasimonotone*, that is, monotone in only the highest order terms.

Let $\mathcal{A} : V \rightarrow V'$ be a function, V a separable reflexive Banach space. We list some examples of conditions which imply $\mathcal{A}$ is type M , that is, if $u_n \rightharpoonup u$ in V , $\mathcal{A}(u_n) \rightharpoonup f$ in V' and $\limsup \mathcal{A}u_n(u_n) \leq f(u)$, then $\mathcal{A}(u) = f$.

EXAMPLE 2.A. Let $\mathcal{A}$ be *weakly closed*: $u_n \rightharpoonup u$ and $\mathcal{A}u_n \rightharpoonup f$ imply $\mathcal{A}u = f$. Clearly, $\mathcal{A}$ is then type M . Note that “bounded and weakly closed” is equivalent to “weakly continuous” in this space. Thus every continuous and linear operator from V to V' is type M ; this holds without any monotonicity.

EXAMPLE 2.B. Type M plus *completely continuous* is type M : if $\mathcal{A}$ is type M and if B is completely continuous, then $\mathcal{A} + B$ is type M . This is immediate since $u_n \rightharpoonup u$ implies $Bu_n \rightarrow Bu$ and so $Bu_n(u_n) \rightarrow Bu(u)$. Thus the type M property is stable under “compact perturbations” in this setting.

EXAMPLE 2.C. Type M plus *monotone, weakly continuous* is type M . Suppose $\mathcal{A}$ is type M and B is monotone, weakly continuous. Let $u_n \rightharpoonup u$, $(\mathcal{A} + B)(u_n) \rightharpoonup f$ and $\limsup (\mathcal{A} + B)(u_n)(u_n) \leq f(u)$. Since B is weakly continuous, $Bu_n \rightharpoonup Bu$ and so $\mathcal{A}u_n \rightharpoonup f - B(u)$. Since B is monotone we have $\mathcal{A}(u_n)(u_n) \leq \mathcal{A}(u_n)(u) + (\mathcal{A} + B)(u_n)(u_n - u) - Bu(u_n - u)$, so $\limsup \mathcal{A}u_n(u_n) \leq (f - Bu)(u)$. But $\mathcal{A}$ is type M , so $\mathcal{A}(u) = f - B(u)$.

Even with monotonicity it is difficult to relax the severe condition of weak continuity of the perturbation in the above. The next example shows that even Lipschitz continuity is not sufficient.

EXAMPLE 2.D. Let $V = V'$ be separable Hilbert space with orthonormal basis $\{e_n : n \geq 1\}$. Set $\mathcal{A} = -I$, the negative of the identity, and P the projection onto the unit ball $K = \{v \in V : \|v\| \leq 1\}$. Recall from I.2 that this projection is characterized by

$$Pu \in K : (Pu - u, Pu - v) \leq 0 \quad \forall v \in K$$

Also, we note that for $u, v \in V$

$$\begin{aligned} (Pu - Pv, u - v) &= (Pu - Pv, u - Pu) + \|Pu - Pv\|^2 + (Pu - Pv, Pv - v) \\ &\geq \|Pu - Pv\|^2, \end{aligned}$$

so P is a monotone contraction.

Choose $u_n = e_1 + e_n$, $n \geq 2$. Then $Pu_n = \frac{1}{\sqrt{2}}u_n$, $(A + P)(u_n) = (\frac{1}{\sqrt{2}} - 1)u_n$ so $u_n \rightharpoonup e_1$, $(A + P)(u_n) \rightharpoonup (\frac{1}{\sqrt{2}} - 1)e_1 \equiv f$, and $((A + P)(u_n, u_n)) = 2(\frac{1}{\sqrt{2}} - 1) < (f, e_1)$. However, $(A + P)(e_1) = 0 \neq f$, so we see that $A + P$ is not type M .

Given $\mathcal{A} : V \rightarrow V'$ and $f \in V'$ as above, suppose we wish to solve the *variational inequality* (cf. I.2.12)

$$(2.2) \quad u \in K : \mathcal{A}u(v - u) \geq f(v - u) , \quad v \in K ,$$

where K is a closed convex subset of V . In order to retain information on the value of the left side of this inequality when we take weak limits, for example, in a Galerkin approximation procedure as in the proof of Theorem 2.1, we introduce the following class of operators.

DEFINITION. Let V be a reflexive Banach space. An operator $\mathcal{A} : V \rightarrow V'$ is *pseudo-monotone* if $u_n \rightharpoonup u$ and $\limsup \mathcal{A}u_n(u_n - u) \leq 0$ imply $\mathcal{A}u(u - v) \leq \liminf \mathcal{A}u_n(u_n - v)$ for all $v \in V$.

Note in the above that from $v = u$ we obtain $\lim \mathcal{A}u_n(u_n - u) = 0$ and so it follows that $\mathcal{A}u(u - v) \leq \liminf \mathcal{A}u_n(u - v)$ for all $v \in V$.

PROPOSITION 2.3. *If $\mathcal{A}$ is monotone and hemicontinuous then $\mathcal{A}$ is pseudo-monotone. If $\mathcal{A}$ is pseudo-monotone, then $\mathcal{A}$ is of type M.*

PROOF. Suppose $u_n \rightharpoonup u$ and $\limsup \mathcal{A}u_n(u_n - u) \leq 0$. By monotonicity, $\mathcal{A}u_n(u_n - u) \geq \mathcal{A}u(u_n - u) \rightarrow 0$ so $\liminf \mathcal{A}u_n(u_n - u) \geq 0$; thus $\lim \mathcal{A}u_n(u_n - u) = 0$. Let $v \in V$ and set $w = (1 - t)u + tv$, $t > 0$. Then $u_n - w = t(u - v) + (u_n - u)$ and by monotonicity $0 \leq \langle \mathcal{A}u_n - \mathcal{A}w, u_n - w \rangle$; this implies

$$t\mathcal{A}u_n(u - v) \geq -\mathcal{A}u_n(u_n - u) + \mathcal{A}w(t(u - v) + (u_n - u)) .$$

Taking the $\liminf$ and dividing by t gives

$$\liminf \mathcal{A}u_n(u - v) \geq \mathcal{A}w(u - v) .$$

Now let $t \downarrow 0$ and use hemicontinuity to get

$$\liminf \mathcal{A}u_n(u - v) \geq \mathcal{A}u(u - v) .$$

Since $\lim \mathcal{A}u_n(u_n - u) = 0$, we are done.

For the second part, let $u_n \rightharpoonup u$, $\mathcal{A}u_n \rightharpoonup f$ and $\limsup \mathcal{A}u_n(u_n) \leq f(u)$. Then for any $v \in V$ we have

$$\begin{aligned} \mathcal{A}u(u - v) &\leq \liminf \mathcal{A}u_n(u_n - v) \leq \limsup \mathcal{A}u_n(u_n - v) \\ &\leq f(u - v) \end{aligned}$$

so it follows that $\mathcal{A}u = f$. □

COROLLARY 2.3. *If $\mathcal{A}$ is pseudo-monotone and locally bounded then it is demicontinuous.*

PROOF. This is immediate from Lemma 2.2, but we can easily prove it directly. If $u_n \rightarrow u$ then (for some subsequence) we have $\mathcal{A}(u_n) \rightharpoonup f$ in V' . Thus, $\mathcal{A}u_n(u_n - u) \rightarrow 0$, so from the definition of pseudomonotone we obtain

$$\mathcal{A}u(u - v) \leq \liminf_{n \rightarrow \infty} \mathcal{A}u_n(u_n - v) = f(u - v) , \quad v \in V ,$$

and this implies $\mathcal{A}u = f$. This holds for any subsequence, so we conclude $\mathcal{A}u_n \rightharpoonup \mathcal{A}u$ by the uniqueness of weak limits. □

PROPOSITION 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be pseudo-monotone; then $\mathcal{A} + \mathcal{B}$ is pseudo-monotone.*

PROOF. Assume $u_n \rightharpoonup u$ and $\limsup (A + B)(u_n)(u_n - u) \leq 0$. We claim that $\limsup \mathcal{A}u_n(u_n - u) \leq 0$ and $\limsup \mathcal{B}u_n(u_n - u) \leq 0$. Otherwise, by symmetry $\limsup \mathcal{B}u_n(u_n - u) = \varepsilon > 0$ and, by passing to a subsequence, $\lim \mathcal{B}u_n(u_n - u) = \varepsilon$. This gives $\limsup \mathcal{A}u_n(u_n - u) = \limsup \{(A + B)u_n(u_n - u) - \mathcal{B}u_n(u_n - u)\} \leq 0 - \varepsilon$. Since $\mathcal{A}$ is pseudo-monotone we have $\mathcal{A}u(u - v) \leq \liminf \mathcal{A}u_n(u_n - v)$ for all $v \in V$; setting $v = u$ gives

$$0 \leq \liminf \mathcal{A}u_n(u_n - u) \leq \limsup \mathcal{A}u_n(u_n - u) \leq -\varepsilon ,$$

a contradiction. With the claim established, the proof now follows by the super-additivity of the “lim inf”. $\square$

We have shown the class of pseudo-monotone operators is “intermediate” between monotone, hemicontinuous and type M , and that it is stable with respect to addition. Moreover, we note that this class is “strictly intermediate”. It is easy to construct pseudo-monotone examples which are not monotone; in Example 2.D we had $(A + P)u_n(u_n - e_1) = \frac{1}{\sqrt{2}} - 1 < 0$, so $A + P$ is not pseudo-monotone. By Propositions 2.3 and 2.4 it follows that $\mathcal{A} = -I$ is not pseudo-monotone, but this is easy to see directly. In fact, $-I$ is pseudo-monotone in Hilbert space V if and only if $\dim(V) < \infty$. In general, for Hilbert space we have $-I$ pseudo-monotone if and only if weak convergence is equivalent to strong convergence.

THEOREM 2.3 (BREZIS). *Let V be a separable reflexive Banach space, K a closed, convex non-empty subset of V , $\mathcal{A} : V \rightarrow V'$ a bounded pseudo-monotone operator, and $f \in V'$. Assume there is a $v_0 \in K$ and $\rho > 0$ such that*

$$(2.3) \quad \mathcal{A}v(v - v_0) > f(v - v_0) \quad \forall v \in K, \quad \|v\| \geq \rho .$$

Then there exists a solution of the variational inequality (2.2).

PROOF. We proceed in four steps. Assume K is bounded, so (2.3) is vacuously true, and (i) solve the finite-dimensional problem by Brouwer’s fixed-point theorem, (ii) use boundedness of $\mathcal{A}$ to get an appropriate subsequence of “approximate” solutions, then (iii) use the pseudo-monotone property to show the weak limit is a solution. Finally, (iv) use (2.3) to eliminate the assumption that K is bounded.

First, let $\{w_1, w_2, \dots\}$ be dense in K , V_m be the linear span of $\{w_1, \dots, w_m\}$, and K_m be the convex hull of $\{w_1, \dots, w_m\}$ for $m \geq 1$. Let $j_m : V_m \rightarrow V$ and its dual $j'_m : V' \rightarrow V'_m$ denote injection and restriction, respectively. Note that $\cup\{K_m : m \geq 1\}$ is dense in K . Fix $m \geq 1$ and consider the finite-dimensional problem

$$u_m \in K_m : j'_m \mathcal{A}j_m u_m(v - u_m) \geq j'_m f(v - u_m) , \quad v \in K_m .$$

This is equivalent to

$$u_m \in K_m : (u_m, v - u_m) \geq (u_m + j'_m f - j'_m \mathcal{A}j_m u_m, v - u_m) , \quad v \in K_m$$

where we identify $V_m \cong \mathbb{R}^m \cong V'_m$, and this is equivalent to

$$u_m = P(u_m + j'_m f - j'_m \mathcal{A}j_m u_m)$$

where P is projection onto K_m in $\mathbb{R}^m$. But the function $u \mapsto P(u + j'_m f - j'_m \mathcal{A} j_m u)$ is continuous from any closed ball B containing K_m into B so the Brouwer fixed-point theorem shows this equation has a solution.

Since we assumed K is bounded, by passing to a subsequence, denoted by $\{u_m\}$, we have $u_m \rightharpoonup u \in K$, and $\|\mathcal{A}(u_m)\| \leq M$ since $\mathcal{A}$ is bounded. Let's show $\limsup \mathcal{A}u_m(u_m - u) \leq 0$. For any $\varepsilon > 0$ choose $\tilde{u} \in K_N$ with $\|u - \tilde{u}\| < \varepsilon$. Then $\mathcal{A}u_m(u_m - \tilde{u}) \leq f(u_m - \tilde{u})$ for $m \geq N$, since $K_N \subset K_m$, so we obtain

$$\limsup \mathcal{A}u_m(u_m - u) = \limsup \{\mathcal{A}u_m(u_m - \tilde{u}) + \mathcal{A}u_m(\tilde{u} - u)\} \leq (\|f\| + M)\varepsilon .$$

But $\varepsilon > 0$ is arbitrary so the claim is done.

Since $\mathcal{A}$ is pseudo-monotone it follows that $\mathcal{A}u(u - v) \leq \liminf \mathcal{A}u_m(u_m - v)$ for all $v \in V$. If $v \in K_n$ and $m \geq n$, then $\mathcal{A}u_m(u_m - v) \leq f(u_m - v)$ so $\mathcal{A}u(u - v) \leq f(u - v)$ for all $v \in K_n$, $n \geq 1$. But $\cup\{K_n : n \geq 1\}$ is dense in K , so u is a solution of (2.2).

It remains only to remove the assumption that K is bounded. Set $R = \max\{\|v_0\|, \rho\}$ and $K_R = \{v \in K : \|v\| \leq R\}$. Since K_R is bounded there is a

$$u_R \in K_R : \mathcal{A}(u_R)(v - u_R) \geq f(v - u_R) , \quad v \in K_R .$$

Since $v_0 \in K_R$ it follows $\mathcal{A}(u_R)(u_R - v_0) \leq f(u_R - v_0)$ so (2.3) implies $\|u_R\| < \rho$. For any $v \in K$ set $v_t \equiv (1 - t)u_R + tv$ with $t > 0$ so small that $v_t \in K_R$. Observe $v_t - u_R = t(v - u_R)$ so it follows that u_R is a solution of the original problem on K and the proof is finished. $\square$

For the special class of monotone operators, we have a *weak characterization* of solutions as well as additional properties of the set of solutions.

COROLLARY 2.4. *If $\mathcal{A}$ is monotone, hemicontinuous, then u is a solution of the variational inequality (2.2) if and only if*

$$(2.4) \quad u \in K : \mathcal{A}v(v - u) \geq f(v - u) , \quad v \in K .$$

If also $\mathcal{A}$ is bounded and satisfies (2.3) then the set of solutions is closed, convex, non-empty and bounded.

PROOF. Since $\mathcal{A}$ is monotone, every solution satisfies (2.4). Conversely, if u satisfies (2.4) and $v \in K$, we choose $0 < t < 1$ and $v_t = u + t(v - u) = (1 - t)u + tv \in K$ as a test vector to obtain $v_t - u = t(v - u)$, hence,

$$\mathcal{A}(v_t)(v - u) \geq f(v - u) .$$

Now use hemicontinuity to let $t \rightarrow 0^+$; this shows the problems are equivalent. Clearly the solution set is convex and closed, since the set of $u \in K$ satisfying (2.4) for a fixed v is closed and convex and we may take intersections. Also (2.3) implies it is non-empty and bounded. $\square$

We mention some easy but useful remarks. The condition (2.3) implies that any solution of the variational inequality satisfies the explicit bound $\|u\| < \rho$. Thus (2.3) is an *a-priori estimate* on solutions. It is vacuously true if K is bounded; if $\mathcal{A}$ is coercive it holds for *every* $f \in V'$. Finally, if $\mathcal{A}$ is strictly monotone there is at most one solution of the variational inequality.

II.3. L^p Spaces

Let G be an open (or Lebesgue measurable) set in $\mathbb{R}^n$ and $1 \leq p < \infty$. Then $L^p(G)$ is the set of equivalence classes of measurable functions $u : G \rightarrow \mathbb{R}$ such that $\int_G |u(x)|^p dx < \infty$, where we identify functions whose values equal a.e. on G . From the (crude) inequality, $|a + b|^p \leq 2^p(a^p + b^p)$ for $a, b \geq 0$, it follows that L^p is a linear space. The space $L^\infty(G)$ consists of all (equivalence classes of) *essentially bounded* measurable functions, $u : G \rightarrow \mathbb{R}$, for each of which there is a $K < \infty$ with $|u(x)| \leq K$ a.e. $x \in G$. The infimum of such K is denoted by $\|u\|_{L^\infty(G)}$; clearly $L^\infty(G)$ is a linear space.

We develop some fundamental inequalities.

LEMMA 3.1 (CAUCHY). $ab \leq \frac{\varepsilon a^2}{2} + \frac{b^2}{\varepsilon 2}$ for $a, b \geq 0$.

PROOF. $(\sqrt{\varepsilon} a - \frac{1}{\sqrt{\varepsilon}} b)^2 \geq 0$. □

LEMMA 3.2 (JENSEN). If $f : G \rightarrow \mathbb{R}$ is integrable and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is convex, then

$$\varphi\left(\frac{1}{\mu(G)} \int_G f(x) dx\right) \leq \frac{1}{\mu(G)} \int_G \varphi(f(x)) dx.$$

PROOF. Set $a = \frac{1}{\mu(G)} \int_G f$ and choose $m \in \mathbb{R} : m(t - a) + \varphi(a) \leq \varphi(t)$ for all $t \in \mathbb{R}$. (That is, m is the slope of a line of support of the graph of φ .) Set $t = f(x)$ and integrate over G . □

LEMMA 3.3 (YOUNG). $ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$ for $a, b \geq 0$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

PROOF. Set

$$f(x) = \begin{cases} p \log a, & 0 \leq x \leq \frac{1}{p} \\ p' \log b, & \frac{1}{p} \leq x \leq 1 \end{cases}, \quad G = (0, 1), \quad \varphi(t) = e^t. \quad \square$$

LEMMA 3.3A. $ab \leq \frac{\varepsilon a^p}{p} + \frac{\varepsilon^{1-p'} b^{p'}}{p'}$.

LEMMA 3.4 (HÖLDER). $\int_G |u(x)v(x)| dx \leq \|u\|_{L^p} \|v\|_{L^{p'}}$, $1 \leq p, p', \frac{1}{p} + \frac{1}{p'} = 1$, where

$$\|u\|_{L^p(G)} \equiv \left(\int_G |u(x)|^p dx \right)^{1/p} \quad \text{for } 1 \leq p < \infty.$$

PROOF. The cases $p = 1, +\infty$ are immediate. Otherwise,

$$\int |uv| dx \leq \frac{\varepsilon}{p} \|u\|_{L^p}^p + \frac{1}{\varepsilon^{p'-1} p'} \|v\|_{L^{p'}}^{p'}$$

and we set $\varepsilon = \|v\|_{L^{p'}} / \|u\|_{L^p}^{p-1}$. □

LEMMA 3.5 (MINKOWSKI). $\|u + v\|_{L^p} \leq \|u\|_{L^p} + \|v\|_{L^p}$.

PROOF. For $1 < p < \infty$,

$$\int |u + v|^{p-1} |u + v| \leq \left(\int |u + v|^{(p-1)p'} \right)^{1-\frac{1}{p}} (\|u\|_{L^p} + \|v\|_{L^p}) .$$

The cases $p = 1, \infty$ are easy. □

Here is an *interpolation* inequality.

LEMMA 3.6. Let $1 \leq p \leq r \leq q$, $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$ and $0 \leq \alpha \leq 1$. Then

$$\|u\|_{L^r} \leq \|u\|_{L^p}^\alpha \|u\|_{L^q}^{1-\alpha} .$$

PROOF.

$$\begin{aligned} \|u\|_{L^r}^r &= \int |u|^{\alpha r + (1-\alpha)r} \\ &\leq \left(\int |u|^{\alpha r \frac{p}{\alpha r}} \right)^{\frac{\alpha r}{p}} \left(\int |u|^{(1-\alpha)r \frac{q}{(1-\alpha)r}} \right)^{\frac{(1-\alpha)r}{q}} \\ &= \|u\|_{L^p}^{\alpha r} \|u\|_{L^q}^{(1-\alpha)r} \end{aligned}$$

since $\frac{\alpha r}{p} + \frac{(1-\alpha)r}{q} = 1$. □

The next inequality is an *imbedding* result.

LEMMA 3.7. If $\mu(G) < \infty$, $1 \leq p \leq q \leq \infty$ then $L^q \subset L^p$ and

$$\|u\|_{L^p} \leq \mu(G)^{\frac{1}{p} - \frac{1}{q}} \|u\|_{L^q} .$$

PROOF.

$$\|u\|_{L^p}^p = \int_G |u(x)|^p \cdot 1 \, dx \leq \|u\|_{L^q}^{p/q} \left(\int_G 1 \right)^{1-\frac{p}{q}} .$$
□

The next result describes the intersection of the L^p spaces.

LEMMA 3.8. If $u \in L^p$, for $p \geq 1$, then $\lim_{q \rightarrow \infty} \|u\|_{L^q} = \|u\|_{L^\infty}$.

PROOF. If $0 < m < \|u\|_{L^\infty}$ and $A_m = \{x : |u(x)| > m\}$, then $\|u\|_{L^q} \geq m \mu(A_m)^{1/q}$ and $0 < \mu(A_m) < \infty$. Therefore $\lim_{q \rightarrow \infty} \|u\|_{L^q} \geq m$, hence $\lim_{q \rightarrow \infty} \|u\|_{L^q} \geq \|u\|_{L^\infty}$. If $\|u\|_{L^\infty} < \infty$ then $|u(x)|^q \leq |u(x)|^p \|u\|_{L^\infty}^{q-p}$ so $\|u\|_{L^q} \leq \|u\|_{L^p}^{p/q} \|u\|_{L^\infty}^{1-(p/q)}$ so $\lim_{q \rightarrow \infty} \|u\|_{L^q} \leq \|u\|_{L^\infty}$. □

PROPOSITION 3.1. $L^p(G)$ is a Banach space, $1 \leq p \leq \infty$.

PROOF. For $p = +\infty$, $\{u_n\}$ Cauchy in $L^\infty(G)$ implies there is a set A of zero measure such that for $x \in G \sim A$, $m, n \geq 1$

$$|u_n(x) - u_m(x)| \leq \|u_n - u_m\|_{L^\infty} , \quad |u_n(x)| \leq \sup_m \|u_m\|_{L^\infty} .$$

Thus $u_n \rightarrow u$ uniformly on $G \sim A$; set $u(x) = 0$ for $x \in A$. Then $\|u\|_{L^\infty} \leq \sup \|u_n\|_{L^\infty}$ and $\|u_n - u\|_{L^\infty} \rightarrow 0$ as $n \rightarrow \infty$.

Suppose $1 \leq p < \infty$ and $\{u_n\}$ is Cauchy in L^p . There is a subsequence $\{u_{n_j}\}$ for which $\|u_{n_{j+1}} - u_{n_j}\|_{L^p} < \frac{1}{2^j}$. Set $v_m(x) \equiv \sum_{j=1}^m |u_{n_{j+1}}(x) - u_{n_j}(x)|$, so $\|v_m\|_{L^p} \leq \sum_{j=1}^m \frac{1}{2^j} < 1$, and $v(x) = \lim_{m \rightarrow \infty} v_m(x)$ in $\mathbb{R}_\infty$. From Fatou's lemma,

$$\int_G |v(x)|^p dx \leq \liminf_{m \rightarrow \infty} \int_G |v_m(x)|^p dx \leq 1 \quad \text{so } v(x) < \infty \text{ a.e.}$$

and the series

$$\left\{ u_{n_1}(x) + \sum_{j=1}^{\infty} (u_{n_{j+1}}(x) - u_{n_j}(x)) \right\}$$

is (absolutely) convergent to (definition) $u(x) \equiv \lim_{j \rightarrow \infty} u_{n_j}(x)$, a.e. on G . Set $u(x) = 0$ otherwise. Let $\varepsilon > 0$ and choose N : $m, n \geq N$ implies $\|u_n - u_m\|_{L^p} < \varepsilon$. Fatou's lemma shows for $n \geq N$ that

$$\begin{aligned} \int_G |u(x) - u_n(x)|^p dx &= \int_G \lim_{j \rightarrow \infty} |u_{n_j}(x) - u_n(x)|^p \\ &\leq \liminf_{j \rightarrow \infty} \int_G |u_{n_j}(x) - u_n(x)|^p dx \leq \varepsilon^p \end{aligned}$$

so $u - u_n \in L^p$, (hence, $u \in L^p$), and $u_n \rightarrow u$ in L^p . $\square$

The proof of Proposition 3.1 also yields the following useful result.

COROLLARY 3.1. *Every Cauchy sequence in $L^p(G)$ has a subsequence which converges pointwise a.e. on G .*

We consider the approximation of functions in $L^p(G)$ by those which are smooth and vanish near the boundary, ∂G . For any real-valued function φ on G we define the *support* of φ to be the closure in G of the set $\{x \in G : \varphi(x) \neq 0\}$, and we denote it by $\text{supp}(\varphi)$. For any pair of sets we denote by $K \subset\subset G$ that K is compact, G is open, and $K \subset G$. This implies there is a strictly positive distance between K and ∂G . Finally we define $C_0^\infty(G) = \{\varphi : G \rightarrow \mathbb{R} \mid \varphi \text{ is infinitely differentiable and } \text{supp}(\varphi) \subset\subset G\}$. This linear space is frequently called the space of *test functions* because it is dense in many function spaces over G and it is easy to justify computations on such functions.

In order to construct smooth approximations of general functions, we specify a *regularizing sequence*: for each $\varepsilon > 0$, let $\varphi_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ be given with

$$\varphi_\varepsilon \geq 0, \quad \text{supp } \varphi_\varepsilon \subset \{x \in \mathbb{R}^n : |x| \leq \varepsilon\}, \quad \int \varphi_\varepsilon = 1.$$

Such functions can be constructed in the form $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi_1(x/\varepsilon)$ from a single φ_1 as given above.

Let $f \in L_{\text{loc}}^1(G)$ and define the *mollifier* $\mathcal{M}_\varepsilon$ corresponding to $\{\varphi_\varepsilon\}$ by

$$(\mathcal{M}_\varepsilon f)(x) = f_\varepsilon(x) = \int_{\mathbb{R}^n} f(x-y) \varphi_\varepsilon(y) dy = \int_{\mathbb{R}^n} f(y) \varphi_\varepsilon(x-y) dy,$$

where we have extended f to all of $\mathbb{R}^n$ as zero. From the first integral representation it follows that

$$\text{supp } f_\varepsilon \subset \text{supp } f + \{y : |y| \leq \varepsilon\}.$$

That is, $f_\varepsilon(x) \neq 0$ only if $x \in \text{supp}(f) + \{y : |y| \leq \varepsilon\}$, $\text{supp } f$ is closed and the ε -ball is compact, so their set sum is closed, hence, it contains $\text{supp}(f_\varepsilon)$. From the second

integral representation and Leibnitz' rule, it follows that $f_\varepsilon \in C^\infty(\mathbb{R}^n)$. Thus we have $\mathcal{M}_\varepsilon : L^1_{\text{loc}} \rightarrow C^\infty_0$ and the support of $\mathcal{M}_\varepsilon f$ is at most ε larger than that of f .

PROPOSITION 3.2. (a) If $f \in C_0(G)$, then $f_\varepsilon \rightarrow f$ uniformly. (b) If $f \in L^p(G)$, $1 \leq p < \infty$, then $\|f_\varepsilon\|_{L^p} \leq \|f\|_{L^p}$ and $f_\varepsilon \rightarrow f$ in $L^p(G)$.

PROOF. The proof of (a) follows from the calculation

$$\begin{aligned} |f_\varepsilon(x) - f(x)| &\leq \int_{\mathbb{R}^n} |f(x-y) - f(x)| \varphi_\varepsilon(y) dy \\ &\leq \sup\{|f(x-y) - f(x)| : y \in f, |y| \leq \varepsilon\}. \end{aligned}$$

For (b) the case $p = 1$ follows from

$$\|f_\varepsilon\|_{L^1} \leq \iint |f(x-y)| \varphi_\varepsilon(y) dx dy = \|f\|_{L^1}.$$

To prove this for $p > 1$, let $\psi \in C_0(G)$ and estimate

$$\begin{aligned} \left| \int_{\mathbb{R}^n} f_\varepsilon(x) \psi(x) dx \right| &\leq \iint |f(x-y) \psi(x)| dx \varphi_\varepsilon(y) dy \\ &\leq \int \|f\|_{L^p} \|\psi\|_{L^{p'}} \varphi_\varepsilon(y) dy \end{aligned}$$

and take the supremum over those ψ with $L^{p'}$ -norm at most one. For the convergence proof, first let $\eta > 0$, then choose $g \in C_0(G)$ with $\|f - g\|_{L^p} < \eta$. Then we have $\|f_\varepsilon - g_\varepsilon\|_{L^p} < \eta$ and, hence, $\|f - f_\varepsilon\|_{L^p} < 2\eta + \|g - g_\varepsilon\|_{L^p}$. The last term goes to zero with ε , by part (a), so we are done. $\square$

We consider the dual of $L^p(G)$. For each $v \in L^{p'}(G)$ define $\mathcal{R}_v(u) \equiv \int_G v u$, $u \in L^p(G)$. Then $\mathcal{R}_v \in (L^p)'$ and $\|\mathcal{R}_v\|_{(L^p)'} \leq \|v\|_{L^{p'}}$ follows from Hölder's inequality, Lemma 3.4. We shall show $\mathcal{R} : L^{p'} \rightarrow (L^p)'$ is an isometry.

Suppose $1 < p \leq \infty$ and set $u(x) \equiv |v(x)|^{p'-1} \text{sgn } v(x)$, $x \in G$, where the *sign function* is defined by $\text{sgn } w = 1$ if $w > 0$ and $\text{sgn } w = -1$ if $w < 0$. Then $|u|^p = |v|^{p(p'-1)} = |v|^{p'}$ so $u \in L^p$ and

$$\mathcal{R}_v(u) = \int |v|^{p'} = \|v\|_{L^{p'}}^{p'(\frac{1}{p'} + \frac{1}{p})} = \|v\|_{L^{p'}} \|u\|_{L^p}.$$

Therefore $\|\mathcal{R}_v\|_{(L^p)'} = \|v\|_{L^{p'}}$. Suppose $p = 1$ and $\|v\|_{L^\infty} > 0$. Let $0 < \varepsilon < \|v\|_{L^\infty}$ and choose $A \subset G$ with $0 < \mu(A) < \infty$, $|v(x)| \geq \|v\|_{L^\infty} - \varepsilon$ for all $x \in A$. Set $u(x) = \text{sgn } v(x)$, $x \in A$ and $u(x) = 0$ on $G \sim A$. Then $u \in L^1(G)$ and

$$\mathcal{R}_v(u) = \int_A |v(x)| dx \geq \mu(A)(\|v\|_{L^\infty} - \varepsilon) = \|u\|_{L^1}(\|v\|_{L^\infty} - \varepsilon),$$

so $\|\mathcal{R}_v\|_{(L^1)'} = \|v\|_{L^\infty}$.

This shows the first part of the following; the second is proved using the Radon-Nikodým Theorem.

THEOREM 3.1 (RIESZ). The map $v \mapsto \mathcal{R}_v : L^{p'} \rightarrow (L^p)'$ is an isometry for $1 \leq p \leq \infty$; it is a bijection if $1 \leq p < \infty$.

COROLLARY 3.2. L^p is reflexive if $1 < p < \infty$.

PROOF. Denote by $\mathcal{R}_{p'} : L^p \rightarrow (L^{p'})'$ the Riesz map $\mathcal{R}_{p'}(u)v = \int_G uv dx$, and similarly by $\mathcal{R}_p : L^{p'} \rightarrow (L^p)'$ the map $\mathcal{R}_p v(u) = \int_G vu dx$. The dual of $\mathcal{R}_p$ is $\mathcal{R}'_p : (L^p)'' \rightarrow (L^{p'})'$ given by $\mathcal{R}'_p \varphi(v) = \varphi(\mathcal{R}_p v)$ for $\varphi \in (L^p)''$ and $v \in L^{p'}$.

Set $X = L^p(G)$ so $\varphi : X \rightarrow X''$ is given by $\varphi(u)(\mathcal{R}_p v) \stackrel{(\varphi)}{=} \mathcal{R}_p v(u) = \mathcal{R}_{p'} u(v)$ $\forall u \in L^p, v \in L^{p'}$, hence, $\mathcal{R}'_p(\varphi(w)) = \mathcal{R}_{p'}(u)$ and $\varphi = (\mathcal{R}'_p)^{-1} \circ \mathcal{R}_{p'}$ is onto.

We shall identify $L^p \cong (L^{p'})', 1 < p \leq \infty$, and $(L^p)'' \cong L^p$ for $1 < p < \infty$ as above. Thus certain functions on G are identified with linear functionals on $L^p(G)$ as prescribed above by $\mathcal{R}_p$ and φ , and these are *consistent*.

THEOREM 3.2 (NEMYTSKII). *Let $f : G \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy the Caratheodory conditions*

(i) *$f(\cdot, r) : G \rightarrow \mathbb{R}$ is measurable for each $r \in \mathbb{R}$, and*

(ii) *$f(x, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is continuous for a.e. $x \in G$.*

Let $1 \leq p, q < \infty, k \in L^q(G)$, and assume

$$|f(x, \xi)| \leq c|\xi|^{p/q} + k(x), \quad \text{a.e. } x \in G, \quad \xi \in \mathbb{R}.$$

Then the operator defined by

$$F(u)(x) = f(x, u(x)), \quad \text{a.e. } x \in G, \quad u \in L^p(G)$$

gives $F : L^p(G) \rightarrow L^q(G)$ which is bounded and strongly continuous.

PROOF. We may assume $f(x, r)$ is continuous in r at every $x \in G$. If $u(x) = \lim u_n(x)$ where each u_n is a simple function, then $(Fu)(x) = \lim(Fu_n)(x)$ for $x \in G$. Each $F(u_n)$ is the sum of a countable set of functions $f(x, c_j)\chi_j(x)$ where χ_j is the characteristic function of a measurable set, so each $F(u_n)$ and hence $F(u)$ is measurable. Also $\|F(u)\|_{L^q} \leq c\|u\|_{L^p}^{p/q} + \|k\|_{L^q} = c\|u\|_{L^p}^{p/q} + \|k\|_{L^q}$ so F is bounded from L^p to L^q .

To show F is continuous, let $u_n \rightarrow u$ in L^p and assume for the moment that $\{u_n\}$ has the properties of the subsequence $\{u_{n_j}\}$ in the proof of completeness of L^p above. Since $f(x, u)$ is continuous in u , it follows $\lim_{n \rightarrow \infty} F(u_n)(x) = F(u)(x)$ for a.e. $x \in G$. Also we have

$$\begin{aligned} |F(u_n)(x)| &\leq c|u_n(x)|^{p/q} + k(x) \leq c\left(\sum_{n=1}^{\infty} |u_{n+1}(x) - u_n(x)| + |u_1(x)|\right)^{p/q} + k(x) \\ &\equiv g(x) \text{ and } g \in L^q(G). \end{aligned}$$

From Lebesgue's Theorem we obtain

$$\|F(u_n) - F(u)\|_{L^q}^q = \int \|F(u_n)(x) - F(u)(x)\|^q dx \rightarrow 0$$

so $F(u_n) \rightarrow F(u)$ in L^q . Thus *every* subsequence (of the original sequence) has a subsequence for which $F(u_n)$ converges to $F(u)$ in L^q . $\square$

COROLLARY 3.3. *If f satisfies the Caratheodory conditions and $|f(x, \xi)| \leq c|\xi|^{p-1} + k(x)$ where $k \in L^{p'}$, $1 < p < \infty$ then $F : L^p \rightarrow L^{p'} \cong (L^p)'$ is bounded and continuous. If also $f(x, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is non-decreasing, then $F : L^p \rightarrow (L^p)'$ is monotone.*

PROOF. Choose $q = p'$, $\frac{1}{p} + \frac{1}{p'} = 1$ so $p/p' = p - 1$. Also

$$\langle F(u) - F(v), u - v \rangle = \int_G \left(f(x, u(x)) - f(x, v(x)) \right) (u(x) - v(x)) dx \geq 0,$$

since $(f(x, \xi) - f(x, \eta))(\xi - \eta) \geq 0$ for all $\xi, \eta \in \mathbb{R}$, $x \in G$. $\square$

COROLLARY 3.4. Let $N \geq 1$ and $f : G \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfy Caratheodory conditions as above: $f(x, \xi)$ is continuous in $\xi \in \mathbb{R}^N$ and measurable in $x \in G$. Let $1 \leq p, q < \infty$, $k \in L^q(G)$ and assume

$$|f(x, \xi)| \leq c \sum_{j=1}^N |\xi_j|^{p/q} + k(x), \text{ a.e. } x \in G, \quad \xi \in \mathbb{R}^N.$$

Then $F(u)(x) \equiv f(x, u_1(x), u_2(x), \dots, u_N(x))$, $x \in G$, defines $F : [L^p(G)]^N \rightarrow L^q(G)$ continuous and bounded.

Operators constructed between L^p spaces as above by substitution are referred to as *Nemytskii operators*. As we have seen they are frequently continuous in the strong topology. Things are different in the weak topology.

EXAMPLE 3.A. The sequence $\varphi_n^{(x)} \equiv \sqrt{\frac{2}{\pi}} \sin(nx)$, $n \geq 1$, is an ortho-normal basis for $L^2(0, \pi)$; specifically, $\varphi_n \rightarrow 0$ in $L^2(0, \pi)$. Similarly, $\cos(nx) \rightarrow 0$ so from $\sin^2(x) = \frac{1}{2}(1 - \cos 2x)$ it follows $(\varphi_n)^2 \rightarrow \frac{1}{\pi}$ in $L^2(0, \pi)$. Thus, $u \mapsto u^2$ is not weakly continuous on $L^2(0, \pi)$. Furthermore, $\{\varphi_n\}$ is bounded in $L^p(0, \pi)$, $1 < p < \infty$, $C_0(0, \pi)$ is dense in $L^{p'}(0, \pi) \cong (L^p)'$ and $\int_G \varphi \varphi_n \rightarrow 0$ for all $\varphi \in C_0(0, \pi)$ as above, so $\varphi_n \rightarrow 0$ in every $L^p(0, \pi)$, $1 < p < \infty$, and $(\varphi_n)^2 \rightarrow \frac{1}{\pi}$ in every $L^q(0, \pi)$, $1 < q < \infty$. Hence $u \mapsto u^2$ is *not* weakly continuous $L^p \rightarrow L^q$ for *any* $1 < p, q < \infty$!

We shall give a sufficient condition for a subset of $L^p(\Omega)$ to be compact. Since this is a metric space it follows that a set is compact, i.e., sequentially compact, if and only if it is complete and *totally bounded*. A subset F of the space X is called an ε -net if F is finite and $X = \cup \{B_\varepsilon(a) : a \in F\}$. That is, F is finite and the set of all balls of radius centered at points of F is a covering of the space X . The space X is *totally bounded* if there is an ε -net for each $\varepsilon > 0$.

PROPOSITION 3.3. The space X is compact if and only if it is complete and *totally bounded*.

PROOF. The necessity is straight forward, and we only verify the sufficiency of these criteria. Since X is complete, to show compactness it suffices to show every sequence has a Cauchy subsequence. If we have a sequence $\{x_{11}, x_{12}, x_{13}, \dots\}$ given, then by total-boundedness there is a subsequence $\{x_{21}, x_{22}, x_{23}, \dots\}$, all of whose points lie in one sphere of radius $1/2$ (since there is a $1/2$ -net which contains the original sequence). Likewise there is a subsequence $\{x_{31}, x_{32}, x_{33}, \dots\}$, all of whose points lie in one sphere of radius $1/3$. Continue selection subsequences in this manner and check that the diagonal sequence $\{x_{11}, x_{22}, x_{33}, \dots\}$ is Cauchy.

COROLLARY 3.5. In a complete space, a closed set is compact if and only if it is *totally bounded*.

These notions are particularly useful for establishing criteria for subsets of function spaces to be compact. An example which we use below is the following.

THEOREM 3.3 (ASCOLI). *Let K be a compact set (in a metric space) and let $\mathcal{F}$ be a bounded subset of functions in $C(K, \mathbb{R})$. If $\mathcal{F}$ is uniformly equicontinuous, i.e., for each $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$ if $f \in \mathcal{F}$ and $d(x_1, x_2) < \delta$, then $\mathcal{F}$ is precompact in $C(K, \mathbb{R})$. That is, its closure $\overline{\mathcal{F}}$ is compact.*

THEOREM 3.4 (FRECHET-KOLMOGOROV). *Let G be open in $\mathbb{R}^n$ and $K \subset\subset G$. Let $\mathcal{F}$ be a bounded subset of $L^p(G)$, $1 \leq p < \infty$. Suppose that for every $\varepsilon > 0$ there is a δ , $0 < \delta < \text{dist}(K, \partial G)$ such that*

$$\left(\int_K |f(x+h) - f(x)|^p dx \right)^{1/p} < \varepsilon, \text{ for all } h \in \mathbb{R}^n, |h| < \delta, f \in \mathcal{F}.$$

Then $\mathcal{F}|_K$ is precompact in $L^p(K)$.

PROOF. With no loss of generality, we assume G is bounded. Also, we let $\bar{f}$ denote the zero-extension to $\mathbb{R}^n$ of each function f on G . It follows that $\{\bar{f} : f \in \mathcal{F}\} \equiv \overline{\mathcal{F}}$ is bounded in $L^p(\mathbb{R}^n)$ and in $L^1(\mathbb{R}^n)$.

Fix $n > \frac{1}{\delta}$; then we have with $M_n = \mathcal{M}_{1/n}$

$$|M_n \bar{f}(x) - \bar{f}(x)| \leq \int_{\mathbb{R}^n} |\bar{f}(x-y) - \bar{f}(x)| \varphi_n(y) dy,$$

and writing $\varphi_n = \varphi_n^{1/p} \cdot \varphi_n^{1/p'}$ applying Hölder's inequality, Lemma 3.4, we obtain

$$|M_n \bar{f}(x) - \bar{f}(x)|^p \leq \int_{|y| \leq 1/n} |\bar{f}(x-y) - \bar{f}(x)|^p \varphi_n(y) dy.$$

Thus we have

$$\int_K |M_n \bar{f}(x) - \bar{f}(x)|^p dx \leq \int_{|y| \leq 1/n} \int_G |\bar{f}(x-y) - \bar{f}(x)|^p dx \varphi_n(y) dy < \varepsilon^p.$$

That is,

$$\|M_n \bar{f} - \bar{f}\|_{L^p(K)} < \varepsilon, \quad n \geq \frac{1}{\delta}, \quad \bar{f} \in \overline{\mathcal{F}}.$$

Next consider $\mathcal{F}_n \equiv \{M_n \bar{f}|_K : \bar{f} \in \overline{\mathcal{F}}\}$ for a fixed n . Since φ_n is bounded,

$$\|M_n \bar{f}\|_{L^\infty(\mathbb{R}^n)} \leq \|\varphi_n\|_{L^\infty(\mathbb{R}^n)} \|\bar{f}\|_{L^1(\mathbb{R}^n)}, \quad f \in \mathcal{F},$$

and we have for $x_1, x_2 \in K$

$$|M_n \bar{f}(x_1) - M_n \bar{f}(x_2)| \leq |x_1 - x_2| \|\varphi_n\|_{W^{1,\infty}} \|\bar{f}\|_{L^1},$$

so $\mathcal{F}_n$ is uniformly bounded and equicontinuous. From Ascoli's Theorem 3.3 it follows that $\mathcal{F}_n$ is precompact in $C(K, \mathbb{R})$, hence, in $L^p(K)$.

Finally, let $\varepsilon > 0$ be given and choose $n > 1/\delta$. Since $\mathcal{F}_n$ is precompact in $L^p(K)$ there is a finite collection of balls of radius ε which cover $\mathcal{F}_n$. The set of corresponding balls of radius 2ε then cover $\mathcal{F}|_K$. $\square$

COROLLARY 3.6. *Let G be open in $\mathbb{R}^n$ and $\mathcal{F}$ a bounded subset of $L^p(G)$. Suppose for $\varepsilon > 0$ and compact $K \subset\subset G$ there is a $\delta > 0$, $\delta < \text{dist}(K, \partial G)$, such that*

$$\int_K |f(x+y) - f(x)|^p dx < \varepsilon^p, \quad |h| < \delta, \quad f \in \mathcal{F},$$

and for $\varepsilon > 0$ there is a compact $K \subset\subset G$, such that $\|f\|_{L^p(G \sim K)} < \varepsilon$, $f \in \mathcal{F}$. Then $\mathcal{F}$ is precompact in $L^p(G)$.

PROOF. Given $\varepsilon > 0$, fix K as above so that $\|f\|_{L^p(G \sim K)} < \varepsilon$, $f \in \mathcal{F}$. Thus $\mathcal{F}|_K$ is precompact in $L^p(K)$ and can be covered by a finite collection $B_\varepsilon(f_i)$, $f_i \in L^p(K)$, $1 \leq i \leq N$. Extend these as zero to get $\bar{f}_i \in L^p(G)$. Then it follows that $\mathcal{F} \subset \bigcup_{i=1}^N B_{2\varepsilon}(\bar{f}_i)$. $\square$

We close with a remark on demicontinuity.

LEMMA 3.9. *If $u_n \rightharpoonup u$ in $L^p(G)$, $1 \leq p \leq +\infty$ and $u_n(x) \rightarrow v(x)$ a.e. $x \in G$ then $u = v$.*

PROOF. Let $G_0 \subset G$ with $\mu(G_0) < \infty$. Then Egorov's Theorem implies that for each $\varepsilon > 0$ there is a measurable $A \subset G_0$ with $\mu(G_0 \sim A) < \varepsilon$ and $u_n \rightarrow v$ uniformly on A . For each measurable $B \subset A$, $\int_B u_n \rightarrow \int_B v$ by uniform convergence. Also $\chi_B \in L^{p'}$ since $\mu(B) < \infty$ so by weak convergence $\int_B u_n = \int_B \chi_B \cdot u_n \rightarrow \int_G \chi_B u = \int_B u$, and so $\int_B (u - v) = 0$ for every $B \subset A$. It follows that $u(x) = v(x)$ a.e. in G . $\square$

PROPOSITION 3.4. *If $\{u_n\}$ is bounded in $L^p(G)$, $1 < p < \infty$, and if $u_n(x) \rightarrow v(x)$ a.e. $x \in G$, then $u_n \rightharpoonup v$.*

PROOF. Otherwise there is an $\varepsilon > 0$, $f \in (L^p)'$ and subsequence $\{u_{n_j}\}$ for which $|f(u_{n_j} - v)| \geq \varepsilon$, $j \geq 1$. Pick a further subsequence $u_{n'_j} \rightharpoonup u$ in L^p and note from above $u = v$, a contradiction. $\square$

The point of Proposition 3.4 is that we need only to have the function $F(u) = f(\cdot, u(\cdot))$ to be bounded and to have $f(x, \cdot)$ continuous in order to get $F : L^p \rightarrow L^q$ to be demicontinuous.

II.4. Sobolev Spaces

We shall introduce certain spaces of functions which with their derivatives belong to $L^p(G)$ and describe the sense in which they have values on ∂G . First we construct a space which is very large and contains all its derivatives. Recall that $C_0^\infty(G)$ is dense in every $L^p(G)$, $1 \leq p < \infty$. We call $C_0^\infty(G)$ the space of *test functions* on G .

DEFINITION. $\mathcal{D}^* = C_0^\infty(G)^*$, the algebraic dual of $C_0^\infty(G)$, is the linear space consisting of all linear functionals on $C_0^\infty(G)$. These linear functionals are called *generalized functions* on G .

EXAMPLE 4.A. For each $f \in L_{\text{loc}}^1(G)$ (that is, $f|_K \in L^1(K)$ for each $K \subset\subset G$) we define $\tilde{f} \in \mathcal{D}^*$ by $\tilde{f}(\varphi) = \int_G f\varphi dx$, $\varphi \in C_0^\infty(G)$. Then $f \mapsto \tilde{f}$ is an injection by which we identify $L_{\text{loc}}^1(G) \subset \mathcal{D}^*$.

EXAMPLE 4.B. For each $f \in L^{p'}$, $1 \leq p < \infty$, we have given $\mathcal{R}f \in (L^p)'$ by $\mathcal{R}f(v) = \int_G f v dx$, $v \in L^p(G)$ and have agreed to identify $f \cong \mathcal{R}f$. If we denote by $i : C_0^\infty(G) \rightarrow L^p(G)$ the identity with dense range, and by $i^* : (L^p)' \rightarrow \mathcal{D}^*$ the algebraic dual which is one-to-one (it is just restriction to $C_0^\infty(G)$), then we find $i^*(\mathcal{R}f) = \mathcal{R}f \circ i = \tilde{f}$, so these identifications are *compatible*.

$$\begin{array}{ccc} (L^p)' & \xrightarrow{i^*} & \mathcal{D}^* \\ \uparrow \mathcal{R} & & \uparrow \\ L^{p'} & \longrightarrow & L_{\text{loc}}^1 \end{array}$$

EXAMPLE 4.C. Let $x_0 \in G$ and define $\delta_{x_0} \in \mathcal{D}^*$ by

$$\delta_{x_0}(\varphi) = \varphi(x_0), \quad \varphi \in C_0^\infty(G).$$

Then $\delta_{x_0} \notin L_{\text{loc}}^1$; similarly we can construct $\delta_S(\varphi) = \int_S \varphi ds$ for any lower-dimensional manifold S in G .

Suppose $f \in L_{\text{loc}}^1(G)$ and for a.e. $(x_1, \dots, x_{n-1})$ that $x \mapsto f(x_1, \dots, x_{n-1}, x)$ is absolutely continuous with derivative $D_n f \in L_{\text{loc}}^1$. For each $\varphi \in C_0^\infty(G)$

$$\widetilde{D_n f}(\varphi) \equiv \int_G (D_n f)\varphi = - \int_G f(D_n \varphi) \equiv -\tilde{f}(D_n \varphi) = -D_n^* \tilde{f}(\varphi)$$

where $D_n^* : \mathcal{D}^* \rightarrow \mathcal{D}^*$ is the adjoint of $D_n : C_0^\infty \rightarrow C_0^\infty$. Thus $\widetilde{D_n f} = -D_n^* \tilde{f}$ for all such f and this *forces* the following.

DEFINITION. The partial derivative of the generalized function $T \in \mathcal{D}^*$ in the j^{th} -coordinate direction is the generalized function $\partial_j T \equiv -D_j^* T$; that is,

$$\partial_j T(\varphi) = -T(D_j \varphi) \quad \text{for } \varphi \in C_0^\infty(G).$$

When $T = \tilde{f}$ as above, we call $\partial_j T = \partial_j \tilde{f}$ the *generalized derivative* of f . Of course we rigged this so that $\partial_j \tilde{f} = \widetilde{D_j f}$ so this extended notion is compatible with the usual notion of derivative. Moreover, we can repeatedly differentiate every element of $\mathcal{D}^*$, hence, every $f \in L_{\text{loc}}^1(G)$.

LEMMA 4.1. Let $D = \{u \in L^p(G) : \partial_j u \in L^q(G)\}$, $1 \leq p, q \leq \infty$, and let $\partial_j : D \rightarrow L^q(G)$ be the corresponding linear function. Then ∂_j has closed graph in $L^p \times L^q$.

PROOF. If $u_n \rightarrow u$ in L^p and $\partial_j u_n \rightarrow v$ in L^q , then $\partial_j u_n(\varphi) = -u_n(D_j \varphi) \rightarrow -u(D_j \varphi) = \partial_j u(\varphi)$ as $n \rightarrow \infty$ so $\partial_j u = v$. $\square$

DEFINITION. Let G be a domain in $\mathbb{R}^n$, $1 \leq p \leq \infty$, $0 \leq m = \text{integer}$. $W^{m,p}(G)$ is the linear space of all $u \in L^p(G)$ for which $\partial^\alpha u \in L^p(G)$ for all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ of non-negative integers with order $|\alpha| = \sum_{j=1}^n \alpha_j \leq m$, where $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n}$. With the corresponding norms given by

$$\|u\|_{m,p} = \left(\sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^p(G)}^p \right)^{1/p}, \quad 1 \leq p < \infty \text{ and}$$

$$\|u\|_{m,\infty} = \max_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^\infty(G)},$$

each is the *Sobolev space* of order m in $L^p(G)$.

We digress in order to connect this notion to related ones that frequently occur in the literature. For any pair $u, v \in L_{\text{loc}}^1(G)$, $v = D^\alpha u$ in the *weak sense* and v is called the α^{th} *weak derivative* of u if

$$\int_G u D^\alpha \varphi = (-1)^{|\alpha|} \int_G v \varphi, \quad \varphi \in C_0^\infty(G).$$

This is clearly equivalent to $\tilde{v} = \partial^\alpha \tilde{u}$, so weak and generalized derivatives are equivalent notions.

Certain properties are obvious. The multiple derivative $\partial^\alpha u$ is independent of the order of differentiation, since this is true of $D^\alpha \varphi$. Similarly it follows that $\partial^\alpha \partial^\beta = \partial^{\alpha+\beta}$. Finally, let v be the α^{th} weak derivative of u in $L^1_{\text{loc}}(G)$ as above and consider the mollifier $\mathcal{M}_\varepsilon$ constructed in Section 3. From the definition of derivative we obtain

$$D^\alpha \mathcal{M}_\varepsilon u(x) = \int_G u(y) D^\alpha_x \rho_\varepsilon(x-y) dy =$$

$$(-1)^{|\alpha|} \int_G u(y) D^\alpha_y \rho_\varepsilon(x-y) dy = \mathcal{M}_\varepsilon \partial^\alpha u(x), \quad x \in G, \quad 0 < \varepsilon < \text{dist}(x, \partial G).$$

That is, $\mathcal{M}_\varepsilon \partial^\alpha = D^\alpha \mathcal{M}_\varepsilon$ in the interior of G . Thus, if $v = \partial^\alpha u$ in $L^1_{\text{loc}}(G)$, then by setting $u_n \equiv \mathcal{M}_{\frac{1}{n}} u$ we obtain (with $u = 0$ outside of G) a sequence $u_n \in C^\infty(G)$ such that for any compact $K \subset G$

$$u_n \rightarrow u \text{ and } D^\alpha u_n \rightarrow v \text{ in } L^1(K)$$

as $n \rightarrow \infty$. In this case, v is called the *strong derivative* of u . From Lemma 4.1 it follows that every strong derivative is also a weak or generalized derivative, so all three notions are equivalent.

Finally, we shall show that smooth functions are dense in $W^{m,p}(G)$, $1 \leq p < \infty$, so this space is equivalently obtained as the completion of $C^\infty(G) \cap W^{m,p}(G)$.

THEOREM 4.1 (MEYERS-SERRIN). *If $1 \leq p < \infty$, then $C^\infty(G) \cap W^{m,p}(G)$ is dense in $W^{m,p}(G)$.*

PROOF. For each integer $n \geq 1$ set

$$G_n = \left\{ x \in G : |x| < n \text{ and } \text{dist}(x, \partial G) > \frac{1}{n} \right\}$$

and let $G_0 = G_{-1} = \varphi$. Then the sequence $\Omega_n = G_{n+1} \sim \overline{G}_{n-1}$ is an open cover of G . For each $n \geq 1$ let $\beta_j \in C_0^\infty(\Omega_j)$ with $\beta_j \geq 0$ and $\sum_{j=1}^\infty \beta_j(x) = 1$ for $x \in G$. Such a collection $\{\beta_j\}$ is called a *partition-of-unity* subordinate to the open cover $\{\Omega_j\}$. Let $u \in W^{m,p}(G)$ and $\varepsilon > 0$. Since $\text{supp}(\beta_n u) \subset \Omega_n$, there is an $a_n > 0$ such that the mollifier $M_n \equiv \mathcal{M}_{a_n}$ satisfies

$$\text{supp } M_n(\beta_n u) \subset \bigcup_{j=-1}^1 \Omega_{n+j}$$

and

$$\|M_n(\beta_n u) - \beta_n u\|_{m,p} < \frac{\varepsilon}{2^n}.$$

Define $w = \sum_{n=1}^\infty M_n(\beta_n u)$ and note that for $x \in \Omega_n$ we have

$$w(x) = \sum_{j=-1}^1 M_{n+j}(\beta_{n+j} u).$$

Hence, $w \in C^\infty(\Omega)$ and

$$\|u - w\|_{m,p} \leq \sum_{n=1}^\infty \sum_{j=-1}^1 \|M_{n+j}(\beta_{n+j} u) - \beta_{n+j} u\|_{m,p} < 3\varepsilon,$$

so the desired approximation is achieved. □

The domain G has the *segment property* if for each $x \in \partial G$ there is an open neighborhood N_x and vector $\vec{v} \neq \vec{0}$ such that $z \in \bar{G} \cap N_x$, $0 < t < 1$ imply $z + t\vec{v} \in G$. Such a domain lies on one side (locally) of its $(n-1)$ -dimensional boundary. By mollifier arguments similar to those above, one may prove the following approximation property:

If $1 \leq p < \infty$ then the subspace of restrictions to G of functions in $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{m,p}(G)$.

Here are some elementary properties of the Sobolev spaces.

PROPOSITION 4.1. $W^{m,p}(G)$ is a Banach space.

PROOF. If $\{u_n\}$ is Cauchy in $W^{m,p}$ and $|\alpha| \leq m$ then $\{\partial^\alpha u_n\}$ is Cauchy in L^p , hence, $u_n \rightarrow u$ and $\partial^\alpha u_n \rightarrow v_\alpha$ in $L^p(G)$. Since ∂^α has a closed graph, $v_\alpha = \partial^\alpha u$. $\square$

By considering the map $u \mapsto (\partial^\alpha u)_{|\alpha| \leq m}$ of $W^{m,p}$ into the product space $L_p^N(G) = L^p(G)^N$, N being the number of multi-indices with order $|\alpha| \leq m$, we find that $W^{m,p}$ is isomorphic to a subspace of a product of L^p spaces.

COROLLARY 4.1. If $1 \leq p < \infty$, then $W^{m,p}$ is separable and every $T \in (W^{m,p})'$ is given by

$$T(u) = \sum_{|\alpha| \leq m} \int_G t_\alpha \partial^\alpha u \, dx, \quad u \in W^{m,p}$$

where $\tilde{t} = (t_\alpha)$ belongs to $[L^{p'}]^N$. If $1 < p < \infty$ then $W^{m,p}$ is reflexive.

PROPOSITION 4.2. A sequence $\{u_n\}$ in $W^{m,p}$, $1 < p < \infty$, is weakly convergent if and only if $\{\partial^\alpha u_n\}$ is weakly convergent in L^p for all α , $|\alpha| \leq m$; in this case we have $u_n \rightharpoonup u$ in $W^{m,p}$ if and only if each $\partial^\alpha u_n \rightharpoonup \partial^\alpha u$ in L^p .

PROOF. We may assume all sequences are bounded. If $u_n \rightharpoonup u$ in $W^{m,p}$ then for $\varphi \in C_0^\infty(G)$ we have

$$\partial^\alpha u_n(\varphi) = (-1)^{|\alpha|} u_n(\partial^\alpha \varphi) \longrightarrow (-1)^{|\alpha|} u(\partial^\alpha \varphi) = \partial^\alpha u(\varphi)$$

and $\{\partial^\alpha u_n\}$ is bounded in L^p with $C_0^\infty(G)$ dense in $L^{p'}$, so $\partial^\alpha u_n \rightharpoonup \partial^\alpha u$ in L^p .

Conversely, let $u_n \rightharpoonup u$ and $\partial^\alpha u_n \rightharpoonup v_\alpha$ in L^p ; each ∂^α is weakly closed, so $\partial^\alpha u_n \rightharpoonup \partial^\alpha u$. The result follows from the representation of $(W^{m,p})'$ given in Corollary 4.1. $\square$

The dual of $W^{m,p}(G)$ is generally more than a space of generalized functions on G . Certainly the restriction to $C_0^\infty(G)$ of an element of $(W^{m,p})'$ belongs to $\mathcal{D}^*$; however, this restriction is *not* injective because $C_0^\infty(G)$ is (in general) not dense in $W^{m,p}(G)$. The problem is that $W^{m,p}$ elements may have non-zero *boundary values*, as we shall see below.

DEFINITION. $W_0^{m,p}(G)$ is the closure of $C_0^\infty(G)$ in $W^{m,p}(G)$.

COROLLARY 4.2. If $1 \leq p < \infty$, then every $T \in (W_0^{m,p}(G))'$ is given by $T = \sum_{|\alpha| \leq m} \partial^\alpha f_\alpha$ where each $f_\alpha \in L^{p'}(G)$. Conversely, every linear combination of at most m^{th} -order derivatives of $L^{p'}$ functions is a generalized function in $(W_0^{m,p})'$.

We define the *trace operator*, i.e., restriction to values on the boundary, in the special case of a half-space. Let $G = \mathbb{R}_+^n = \{x = (x', x_n) : x_n > 0\} \subset \mathbb{R}^n$

so $\partial G = \mathbb{R}^{n-1}$. For any $\varphi \in C_0^\infty(\mathbb{R}^n)$ we define $\gamma\varphi = \varphi|_{\partial G}$ and note that the restrictions to G of such functions are dense in $W^{1,p}(G)$, $1 \leq p < \infty$. We compute

$$\begin{aligned} |\varphi(x', 0)|^p &= - \int_0^\infty D_n |\varphi|^p dx_n \leq p \int_0^\infty |\varphi|^{p-1} |D_n \varphi| \quad (p-1 = p/p') \\ &\leq p \|D_n \varphi\|_{L^p(\mathbb{R}^+)} \|\varphi|^{p/p'}\|_{L^{p'}} = p \|D_n \varphi\|_{L^p(\mathbb{R}^+)} \|\varphi\|_{L^p(\mathbb{R}^+)}^{p/p'} , \end{aligned}$$

apply Young's inequality, and integrate over $\mathbb{R}^{n-1}$ to get successively

$$|\varphi(x', 0)|^p \leq \|D_n \varphi(x', \cdot)\|_{L^p(\mathbb{R}_+)}^p + \|\varphi(x', \cdot)\|_{L^p}^p (p-1)$$

$$\|\gamma\varphi\|_{L^p(\mathbb{R}^{n-1})}^p \leq \|D_n \varphi\|_{L^p(\mathbb{R}_+^n)}^p + (p-1) \|\varphi\|_{L^p(\mathbb{R}_+^n)}^p .$$

Thus, the trace operator $\gamma : C_0^\infty(\mathbb{R}^n)|_{\mathbb{R}_+^n} \rightarrow C_0^\infty(\mathbb{R}^{n-1})$ has a unique extension by continuity to

$$\gamma : W^{1,p}(\mathbb{R}_+^n) \longrightarrow L^p(\mathbb{R}^{n-1}) , \quad 1 \leq p < \infty .$$

Note that the formula

$$u(x', x_n) = \int_0^{x_n} \partial_n u(x', s) ds + \gamma u(x') , \quad x' \in \mathbb{R}^{n-1} , \quad x_n > 0$$

extends to $u \in W^{1,p}(\mathbb{R}_+^n)$ from smooth functions.

Let's characterize the kernel of γ , $\ker(\gamma)$. This is a closed subspace of $W^{1,p}(\mathbb{R}_+^n)$ which contains $C_0^\infty(\mathbb{R}_+^n)$. Suppose $u \in \ker(\gamma)$. We shall show there is a sequence in $C_0^\infty(\mathbb{R}_+^n)$ which converges in $W^{1,p}(\mathbb{R}_+^n)$ to u . Pick a sequence $\theta_j : \mathbb{R} \rightarrow \mathbb{R}$ by $\theta_j(t) = 0$, $t \leq 1/j$, $\theta_j(t) = 1$ for $t \geq 2/j$, and $\theta_j(t) = j(t-1/j)$ otherwise, and define $u_j(x', x_n) = \theta_j(x_n)u(x')$. Each u_j is in the closure of $C_0^\infty(\mathbb{R}_+^n)$ (by a convolution approximation) so it suffices to show $\lim_{j \rightarrow \infty} u_j = u$ in $W^{1,p}(\mathbb{R}_+^n)$.

By Lebesgue's dominated convergence theorem it follows that $u_j \rightarrow u$ in L^p and for each k with $1 \leq k \leq n-1$ that $\partial_k(u_j) = \theta_j(\partial_k u) \rightarrow \partial_k u$ in $L^p(\mathbb{R}_+^n)$ as $j \rightarrow \infty$. Similarly, $\theta_j \partial_n u \rightarrow \partial_n u$ and $\partial_n(\theta_j u) = \theta_j(\partial_n u) + \theta_j' u$, so it suffices to show $\theta_j' u \rightarrow 0$ in $L^p(\mathbb{R}_+^n)$. Since $\gamma u = 0$ in $L^p(\mathbb{R}^{n-1})$ our formula above gives

$$u(x', x_n) = \int_0^{x_n} \partial_n u(x', t) dt , \quad x' \in \mathbb{R}^{n-1} , \quad x_n > 0 ,$$

so we compute

$$\begin{aligned} |u(x', x_n)| &\leq \left(\int_0^{x_n} |\partial_n u(x', t)|^p dt \right)^{1/p} x_n^{1/p'} , \quad (\text{Hölder}) \\ \int_0^\infty |\theta_j'(x_n)u(x', x_n)|^p dx_n &\leq \int_0^{2/j} \int_0^{x_n} |\partial_n u(x', t)|^p dt x_n^{p-1} j^p dx_n \\ &= j^p (2/j)^{p-1} \int_0^{2/j} \int_0^{x_n} |\partial_n u(x', t)|^p dt dx_n \\ &= 2^{p-1} j \int_0^{2/j} \int_t^{2/j} |\partial_n u(x', t)|^p dx_n dt \\ &\leq 2^{p-1} j (2/j) \int_0^{2/j} |\partial_n u(x', t)|^p dt . \end{aligned}$$

Integration over $\mathbb{R}^{n-1}$ gives

$$\|\theta'_j u\|_{L^p(\mathbb{R}^{n-1})}^p \leq 2^p \int_0^{2/j} \int_{\mathbb{R}^{n-1}} |\partial_n u|^p dx ,$$

and this converges to zero as $j \rightarrow \infty$. This proves the following.

LEMMA 4.2. *The kernel of $\gamma : W^{1,p}(\mathbb{R}_+^n) \rightarrow L^p(\mathbb{R}^{n-1})$ is the closure in $W^{1,p}(\mathbb{R}_+^n)$ of $C_0^\infty(\mathbb{R}_+^n) : \ker(\gamma) = W_0^{1,p}(\mathbb{R}_+^n)$.*

Finally we shall describe the extension of the trace operator to a bounded open domain $G \subset \mathbb{R}^n$. Assume ∂G is an $n-1$ dimensional C^m manifold. Letting $Q = \{y \in \mathbb{R}^n : |y_i| \leq 1\}$, $Q_0 = \{y \in Q : y_n = 0\}$ and $Q_+ = \{y \in Q : y_n > 0\}$, we state this last condition as follows. There is a collection of open bounded sets $\{G_j : 1 \leq j \leq N\}$ with $\cup\{G_j : 1 \leq i \leq N\} \supset \partial G$ and corresponding $\varphi_j \in C^m(Q, G_j)$ which are bijections of Q , Q_+ and Q_0 onto G_j , $G_j \cap G$ and $G_j \cap \partial G$, respectively, and each Jacobian $J(\varphi_j)$ is positive. (Each pair (φ_j, G_j) is a *coordinate patch*.) Let $G_0 = G$. We can construct $\beta_j \in C_0^\infty(G_j)$, $0 \leq j \leq N$, with $\beta_j(x) \geq 0$, and $\sum_{j=0}^N \beta_j(x) = 1$ for $x \in \overline{G}$. Thus, $\{\beta_j : 1 \leq j \leq N\}$ is a *partition-of-unity* subordinate to the open cover $\{G_j : 1 \leq j \leq N\}$ of ∂G , and $\{\beta_j : 0 \leq j \leq N\}$ is a partition-of-unity subordinate to the open cover $\{G_j : 0 \leq j \leq N\}$ of $\overline{G}$.

If f is a function defined on ∂G then

$$\int_{\partial G} f \equiv \sum_{j=1}^N \int_{\partial G \cap G_j} \beta_j f ds \equiv \sum_{j=1}^n \int_{Q_0} (\beta_j f) \circ \varphi_j(y', 0) J_j(y') dy'$$

where $s = \varphi_j(y', 0)$ and

$$J_j(y') = \left\{ \sum_{k=1}^N \left(\frac{\partial(s_1, \dots, \hat{s}_k, \dots, s_n)}{\partial(y_1, \dots, y_{n-1})} \right)^2 \Big|_{y_n=0} \right\}^{1/2}$$

By the smoothness property, $|J_j(y')| \leq K$, $1 \leq j \leq N$, $y' \in Q_0$, since $m \geq 1$. Finally, we construct the trace on ∂G as indicated.

$$\begin{array}{ccc} W^{1,p}(G) & \longrightarrow & W_0^{1,p}(G) \times W^{1,p}(Q_+)^N \\ u = \sum_{j=0}^N \beta_j u & \longmapsto & \beta_0 u, (\beta_j u) \circ \varphi_j, \quad 1 \leq j \leq N \\ & & \downarrow \\ \gamma(u) \equiv \sum_{j=1}^N \beta_j \gamma_j u & \longleftarrow & \gamma_j(\beta_j u \circ \varphi_j) = (\beta_j \gamma_j(u)) \circ \varphi_j, \quad 1 \leq j \leq N \\ L^p(\partial G) & & L^p(Q_0)^N \end{array}$$

That $\gamma(u) \in L^p(\partial G)$ follows from the estimates

$$\begin{aligned} \int_{\partial G} |\gamma(u)|^p ds &\leq \sum_{j=1}^N \int_{\partial G \cap G_j} |\gamma_j u|^p ds \leq K \sum_{j=1}^N \int_{Q_0} |\gamma_j(u) \circ \varphi_j|_{L^p}^p dy' \\ &\leq K \cdot K_p^p \sum_{j=1}^N \|u \circ \varphi_j\|_{W^{1,p}(Q_+)}^p \leq K \cdot K_p^p K_\varphi^p \sum_{j=1}^N \|u\|_{W^{1,p}(G \cap G_j)}^p \\ &\leq K \cdot K_p^p \cdot K_\varphi^p \cdot N \|u\|_{W^{1,p}(G)}^p, \end{aligned}$$

where K is the maximum of all Jacobians, K_p is the norm of trace from half-space, and K_φ is the largest norm in $W^{1,p}$ under change of variables $\tilde{\varphi}_j : W^{1,p}(Q_+) \rightarrow W^{1,p}(G \cap G_j)$. Clearly, if $u \in C(\bar{G}) \cap W^{1,p}(G)$ then $\gamma(u) = u|_{\partial G}$. We summarize the above as follows.

PROPOSITION 4.3. *Let $1 \leq p < \infty$ and assume G is a bounded domain in $\mathbb{R}^n$ whose boundary ∂G is a C^1 manifold of dimension $n - 1$. Then the linear trace operator $\gamma : W^{1,p}(G) \rightarrow L^p(\partial G)$ is continuous and uniquely determined by $\gamma(u) = u|_{\partial G}$ on those $u \in C^1(\bar{G})$. For any $u \in W^{1,p}(G)$, $\gamma(u) = 0$ in $L^p(\partial G)$ if and only if $u \in W_0^{1,p}(G)$.*

We consider the *extension* of functions in $W^{1,p}(G)$ to all of $\mathbb{R}^n$ in the situation of Proposition 4.3. First, one shows for the special case of a cube, Q , that

$$Eu(x) = \begin{cases} u(x', x_n), & x_n \geq 0 \\ u(x', -x_n), & x_n < 0. \end{cases}$$

determines an extension operator $E : W^{1,p}(Q_+) \rightarrow W^{1,p}(Q)$ for which $E(u)|_{Q_+} = u$, and $\|E(u)\|_{W^{1,p}(Q)} = 2\|u\|_{W^{1,p}(Q_+)}$. We apply this operator to each of the functions $(\beta_j u) \circ \varphi_j \in W^{1,p}(Q_+)$, and note that each $E(\beta_j u \circ \varphi_j)$ has support within Q . Thus we obtain an extension operator $E_G : W^{1,p}(G) \rightarrow W_0^{1,p}(\tilde{G})$ as above where $\tilde{G}$ is any open set containing $\cup\{G_j : 0 \leq j \leq N\}$. Of course, any open $\tilde{G} \supset G$ could have been specified first and then the partition $\{G_i\}$ chosen as before with $G_j \subset \tilde{G}$. This leads to the following.

COROLLARY 4.3. *Let G be given as above and $\tilde{G}$ be open, $\tilde{G} \supset G$. There exists an extension operator $E_G : W^{1,p}(G) \rightarrow W_0^{1,p}(\tilde{G})$ for which $E_G u(x) = u(x)$ for $x \in G$ and*

$$\|E_G(u)\|_{W^{1,p}(\tilde{G})} \leq C(G, \tilde{G}) \|u\|_{W^{1,p}(G)}, \quad u \in W^{1,p}(G).$$

THEOREM 4.2 (RELICH-KANDOROVICH). *Let G be a bounded domain in $\mathbb{R}^n$ and $1 \leq p < \infty$. Then the imbedding $W_0^{1,p}(G) \rightarrow L^p(G)$ is compact. If also the boundary ∂G is a C^1 manifold of dimension $n - 1$, then the imbedding $W^{1,p}(G) \rightarrow L^p(G)$ is compact.*

PROOF. Choose an open set $\tilde{G}$ with $K \equiv \bar{G} \subset \subset \tilde{G}$, $\bar{G}$ denoting the closure of G . For each $\varphi \in C_0^\infty(G)$, the zero-extension belongs to $C_0^\infty(\tilde{G})$, and we denote it by φ . For any $h \in \mathbb{R}^n$ with $|h| < \text{dist}(G, \partial\tilde{G})$ we have for $x \in G$

$$\begin{aligned} \varphi(x+h) - \varphi(x) &= \int_0^1 \frac{d}{dt} \varphi(x+th) dt = \int_0^1 h \cdot \vec{\nabla} \varphi(x+th) dt, \\ |\varphi(x+h) - \varphi(x)|^p &\leq |h|^p \int_0^1 |\vec{\nabla} \varphi(x+th)|^p dt. \end{aligned}$$

Since $G+th \subset \tilde{G}$ for $t \in [0, 1]$, we obtain from a change of order of integration

$$\begin{aligned} \int_G |\varphi(x+h) - \varphi(x)|^p dx &\leq |h|^p \int_0^1 \left\{ \int_{G+th} |\vec{\nabla} \varphi(y)|^p dy \right\} dt \\ &\leq |h|^p \int_{\tilde{G}} |\vec{\nabla} \varphi(y)|^p dy. \end{aligned}$$

This estimate holds for all $\varphi \in W_0^{1,p}(G)$. Let $\mathcal{F}$ be the unit ball in $W_0^{1,p}(\tilde{G})$. From the Frechet-Kolmogorov Theorem 3.4 it follows that $\mathcal{F}|_G$ is compact in $L^p(G)$, but $\mathcal{F}|_G$ is contained in the unit ball in $W_0^{1,p}(G)$. $\square$

For the case of a domain G with smooth boundary, note that the composite operator,

$$W^{1,p}(G) \xrightarrow{E_G} W_0^{1,p}(\tilde{G}) \longrightarrow L^p(\tilde{G}) \longrightarrow L^p(G)$$

consisting of the continuous extension, the compact imbedding, and the continuous restriction, is necessarily compact.

We cite finally the following result from [1, p. 97]. It is very important for many types of nonlinear problems, especially those in which one needs to show that certain products of functions belong to an L^p class. The domain G satisfies the *cone condition* if there is a fixed cone K such that at any point $y \in \partial G$ one can place the vertex at y with $K - y$ lying within G . This is true, for example, in the case of polyhedra and of bounded domains with ∂G being a C^1 manifold.

THEOREM 4.3 (SOBOLEV). *Let G be a bounded domain which satisfies the cone condition in $\mathbb{R}^n$, $1 \leq p < \infty$, and let $m \geq 0$ be an integer.*

If $m < \frac{n}{p}$, then

$$W^{m,p}(G) \rightarrow L^q(G), \quad p \leq q \leq \frac{np}{n-mp}.$$

If $m = \frac{n}{p}$, then the above holds for $p \leq q < \infty$. If $p = 1$, this holds for $q = \infty$, and

$$W^{n,1}(G) \rightarrow C_b(G),$$

where $C_b(G)$ is the space of continuous and bounded functions on G with the sup norm. If $mp > n$, then

$$W^{m,p}(G) \rightarrow C_b(G).$$

Assume ∂G is locally Lipschitz. If $m - 1 < \frac{n}{p} < m$, then

$$W^{m,p}(G) \rightarrow C^\lambda(\bar{G}), \quad 0 < \lambda \leq m - \frac{n}{p},$$

where $C^\lambda(\bar{G})$ is the space of (uniformly) continuous functions on $\bar{G}$ which satisfy a Hölder condition of exponent λ , $0 < \lambda \leq 1$. If $m - 1 = \frac{n}{p}$, then the above holds for all $0 < \lambda < 1$; if also $p = 1$, it holds for $\lambda = 1$.

II.5. Elliptic Boundary-Value Problems

We shall begin with an elementary but instructive model problem, a nonlinear Dirichlet problem. This will be resolved by our general existence theorems applied to operators from a Sobolev space to its dual. This will lead to additional general results as well as applications.

EXAMPLE 5.A. Let G be a domain in $\mathbb{R}^n$, $1 < p < \infty$, and define $a : \mathbb{R} \rightarrow \mathbb{R}$ by $a(\xi) = |\xi|^{p-1} \operatorname{sgn} \xi = |\xi|^{p-2} \xi$. Also let $F \in L^{p'}(G)$, where $1/p + 1/p' = 1$ as usual, and consider the nonlinear *Dirichlet problem* of finding a function $u : G \rightarrow \mathbb{R}$ which satisfies

$$(5.1.a) \quad - \sum_{j=1}^n \partial_j a(\partial_j u(x)) = F(x), \quad x \in G,$$

$$(5.1.b) \quad u(s) = 0, \quad s \in \partial G,$$

in some sense. For example, since $a(\cdot)$ determines a Nemytskii operator from $L^p(G)$ to $L^{p'}(G)$ we seek $u \in W^{1,p}(G)$; then the condition (5.1.b) on ∂G is meaningful in the sense of trace. Finally, the partial differential equation (5.1.a) should hold (at least) in the sense of $\mathcal{D}^*(G)$, i.e.,

$$\sum_{j=1}^n \int_G a(\partial_j u(x)) \partial_j \varphi(x) dx = \int_G F(x) \varphi(x) dx, \quad \varphi \in C_0^\infty(G),$$

and this is equivalent to requiring this identity to hold for all $\varphi \in W_0^{1,p}(G)$.

The linear case ($p = 2$) as discussed in Chapter I suggests that the Dirichlet problem (5.1) can be formulated as an operator equation. Thus, define $V = W_0^{1,p}(G)$ and

$$\mathcal{A}u(v) \equiv \sum_{j=1}^n \int_G a(\partial_j u(x)) \partial_j v(x) dx, \quad u, v \in V.$$

This is meaningful by the estimates

$$\begin{aligned} |\mathcal{A}u(v)| &\leq \sum_{j=1}^n \left(\int_G |\partial_j u|^{\overbrace{p}^{p'(p-1)}} \right)^{1/p'} \left(\int_G |\partial_j v|^p \right)^{1/p} \\ &\leq \left(\sum_{j=1}^n \|\partial_j u\|_{L^p}^p \right)^{1/p'} \left(\sum_{j=1}^n \|\partial_j v\|_{L^p}^p \right)^{1/p} \leq \|u\|_{W_0^{1,p}}^{p-1} \|v\|_{W_0^{1,p}} \end{aligned}$$

from which it follows that $\mathcal{A} : V \rightarrow V'$ is *bounded*. Moreover, $\mathcal{A}$ is the sum of terms which are composition of $\partial_j : W_0^{1,p} \rightarrow L^p$, $a(\cdot) : L^p \rightarrow L^{p'}$, and $\partial_j^* = -\partial_j : L^{p'} \rightarrow (W_0^{1,p})'$, each of which is continuous, so $\mathcal{A} : V \rightarrow V'$ is *continuous*. For a

pair $u, v \in V$ we have

$$\langle \mathcal{A}u - \mathcal{A}v, u - v \rangle = \sum_{j=1}^n \int_G \left(a(\partial_j u(x)) - a(\partial_j v(x)) \right) (\partial_j u(x) - \partial_j v(x)) dx ,$$

and the integrands are non-negative a.e. in G because $a(\cdot)$ is monotone; thus, $\mathcal{A}$ is *monotone* and type M . Our Dirichlet problem (5.1) is of the form $\mathcal{A}u = f$ in V' , where $f \in V'$ is defined by

$$f(v) = \int_G F(x)v(x) dx , \quad v \in V ,$$

and we need only to verify that $\mathcal{A}$ is V -coercive in order to apply our general existence theorems. Note that

$$\mathcal{A}v(v) = \sum_{j=1}^n \|\partial_j v\|_{L^p}^p .$$

LEMMA 5.1 (POINCARÉ). *Assume G is bounded in the first coordinate direction:*

$$\sup\{|x_1| : x = (x_1, \dots, x_n) \in G\} \equiv K < \infty .$$

Then $\|v\|_{L^p(G)} \leq pK \|\partial_1 v\|_{L^p(G)}$ for all $v \in W_0^{1,p}(G)$.

PROOF. It suffices to consider $\varphi \in C_0^\infty(G)$ and note

$$\partial_1 (x_1 |\varphi(x)|^p) = |\varphi(x)|^p + x_1 p |\varphi(x)|^{p-1} \operatorname{sgn}(\varphi(x)) \partial_1 \varphi(x) .$$

Integrating this identity over G and using the divergence theorem yields the estimate

$$\int_G |\varphi(x)|^p dx \leq pK \|\varphi\|_{L^p}^{p-1} \|\partial_1 \varphi\|_{L^p}$$

and, hence, the desired result. $\square$

COROLLARY 5.1. *If G is bounded in some direction, there exists $c > 0$ for which*

$$\sum_{j=1}^n \|\partial_j v\|_{L^p(G)}^p \geq c \|v\|_{W^{1,p}(G)}^p , \quad v \in W_0^{1,p}(G) .$$

We remark that Lemma 5.1 holds if $v \in W^{1,p}(G)$ and $\gamma v(s) = 0$ for those $s \in \partial G$ with $s_1 \nu_1(s) > 0$, where $\vec{\nu} = (\nu_1, \dots, \nu_n)$ is the unit outward normal on ∂G . The problem (5.1) extends easily to the following.

PROPOSITION 5.1 (BROWDER-VISHIK). *Let G be a domain in $\mathbb{R}^n$, bounded in some direction, $1 < p < \infty$. Assume given the functions $A_j : G \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ for $j = 0, 1, \dots, n$ which satisfy*

- (i) $A_j(x, \xi)$ is measurable in x and continuous in ξ ,
- (ii) $|A_j(x, \xi)| \leq c \sum_{j=0}^n |\xi_j|^{p-1} + k(x), \quad \xi \in \mathbb{R}^{n+1}, x \in G$
- (iii) $\sum_{j=0}^n (A_j(x, \xi) - A_j(x, \eta))(\xi_j - \eta_j) \geq 0, \quad \xi, \eta \in \mathbb{R}^{n+1}, x \in G$
- (iv) $\sum_{j=0}^n A_j(x, \xi) \xi_j \geq c_0 \sum_{j=1}^n |\xi_j|^p - k(x), \quad \xi \in \mathbb{R}^{n+1}, x \in G$

where $c_0 > 0$, $k \in L^{p'}(G)$. Let $F_0, F_1, \dots, F_n$ be given in $L^{p'}(G)$. Then the set of solutions to the generalized Dirichlet problem

$$(5.2) \quad \begin{aligned} u \in W_0^{1,p}(G) : \sum_{j=0}^n \int_G A_j(x, u(x), \vec{\nabla} u(x)) \partial_j v(x) dx \\ = \sum_{j=0}^n \int_G F_j(x) \partial_j v(x) dx \text{ for all } v \in W_0^{1,p}(G) \end{aligned}$$

is closed, convex, bounded and non-empty.

PROOF. From (i) and (ii) we define $\mathcal{A} : W_0^{1,p}(G) \rightarrow W_0^{1,p}(G)'$ by the left side of (5.2) and note that it is continuous with bound given by $\|\mathcal{A}(u)\|_{(W_0^{1,p})'} \leq c\|u\|_{W_0^{1,p}}^{p-1} + \|k\|_{L^{p'}}(n+1)^{1/p'}$. That $\mathcal{A}$ is monotone and coercive follow from (iii) and (iv), respectively. $\square$

Non-homogeneous boundary conditions can be attained routinely in the above situation. We need only consider the translation $\mathcal{A}_0(u) = \mathcal{A}(u + u_0)$ where $u_0 \in W_0^{1,p}(G)$ is prescribed. Alternatively, we may set $K = u_0 + W_0^{1,p}(G)$ and use Theorem 2.3. The partial differential equation solved in $\mathcal{D}^*(G)$ in (5.2) is

$$-\sum_{j=0}^n \partial_j A_j(x, u(x), \vec{\nabla} u(x)) = F_0(x) - \sum_{j=1}^n \partial_j F_j(x)$$

which is *quasilinear* with “first-order” distributions on the right-side — rather non-regular and not necessarily even functions. The easiest examples of A_j ’s satisfying (iii) are those which depend only on x, ξ_j , i.e., $A_j(x, \xi) = c_j(x)a_j(\xi_j)$ with $0 \leq c_j \in L^\infty(G)$ and $a_j : \mathbb{R} \rightarrow \mathbb{R}$ monotone and continuous. We shall relax these implicit restrictions in the class of *quasimonotone* operators in Section 6.

Finally, note that if the inequality in (iii) is strict for $\xi \neq \eta$, then whenever $u, v \in W_0^{1,p}(G)$ with $\mathcal{A}u = \mathcal{A}v$, hence, $\langle \mathcal{A}(u) - \mathcal{A}(v), u - v \rangle = 0$, then

$$\sum_{j=0}^n \left(A_j(x, u(x), \vec{\nabla} u(x)) - A_j(x, v(x), \vec{\nabla} v(x)) \right) \partial_j (u(x) - v(x)) = 0, \quad \text{a.e. } x \in G,$$

hence, $(u(x), \vec{\nabla} u(x)) = (v(x), \nabla v(x))$ in $\mathbb{R}^{n+1}$ for a.e. x . That is, there is *at most* one solution. But this rather special restriction can be considerably relaxed!

We pass on to different boundary-value problems for similar quasilinear equations. For simplicity we consider the following special case:

Let G be a bounded domain in $\mathbb{R}^n$ with ∂G a C^1 manifold, $1 < p < \infty$, and $a(\xi) = |\xi|^{p-1} \text{sgn } \xi$, $b(\cdot) \in L^\infty(G)$, $c(\cdot) \in L^\infty(\partial G)$, both $b(\cdot)$ and $c(\cdot)$ are non-negative, and define

$$(5.3) \quad \begin{aligned} \mathcal{A}u(v) = \int_G \left\{ \sum_{j=1}^n a(\partial_j u(x)) \partial_j v(x) + b(x)a(u(x))v(x) \right\} dx \\ + \int_{\partial G} c(s)a(u(s))v(s) ds, \quad u, v \in W^{1,p}(G). \end{aligned}$$

Here we denote $u(s) = (\gamma u)(s)$, where $\gamma : W^{1,p}(G) \rightarrow L^p(\partial G)$ is the trace operator. Let $F \in L^{p'}(G)$, $F_0 \in L^{p'}(\partial G)$ and set

$$f(v) = \int_G F(x)v(x) dx + \int_{\partial G} F_0(s)\gamma v(s) ds, \quad v \in W^{1,p}(G).$$

Although $\mathcal{A}$, f are defined on all of $W^{1,p}(G)$, we shall frequently consider their restrictions to a given closed subspace V , with $W_0^{1,p}(G) \subset V \subset W^{1,p}(G)$. The extreme case $V = W_0^{1,p}$ (with $b = 0$, $c = 0$, $F_0 = 0$) gave us the original model problem, the *Dirichlet problem* (5.1) known as the boundary-value problem of *first type*. In this case, the boundary conditions are obtained from the inclusion, $u \in V$, and these are called the *stable boundary conditions*. That is, they are imposed by the definition of the closed subspace V .

EXAMPLE 5.B. Choose $V = W^{1,p}(G)$, the other extreme case. Then $\mathcal{A} : V \rightarrow V'$ is monotone, continuous and bounded; if $b(x) \geq b_0 > 0$ then $\mathcal{A}$ is strictly-monotone and V -coercive. Thus, there is a unique

$$(5.4) \quad u \in V : \mathcal{A}u(v) = f(v), \quad v \in V,$$

and we shall characterize it as a partial differential equation and boundary condition. Note that there are no boundary constraints imposed in this case by the inclusion, $u \in V$. First, since $C_0^\infty(G) \subset V$ we obtain

$$-\sum_{j=1}^n \partial_j a(\partial_j u) + b(x)a(u) = F$$

in $L^{p'}(G) \subset \mathcal{D}^*(G)$. Substituting this back into the functional equation yields

$$\int_G \left\{ \sum_{j=1}^n a(\partial_j u) \partial_j v + \sum_{j=1}^n \partial_j (a(\partial_j u)) v \right\} + \int_{\partial G} (c(s)a(u) - F_0) v ds = 0, \quad v \in V$$

Now (formally, unless u is smoother) this first integrand is $\sum_{j=1}^n \partial_j (a(\partial_j u)) v$ and the divergence theorem gives

$$\int_{\partial G} \left(\sum_{j=1}^n a(\partial_j u) \nu_j + c(s)a(u) - F_0 \right) \gamma v(s) ds = 0, \quad v \in V.$$

But $Rg(\gamma)$ is dense in $L^p(\partial G)$, so it follows that a (sufficiently smooth) solution of (5.4) is characterized by

(5.5.a)

$$u \in W^{1,p}(G) : -\sum_{j=1}^n \partial_j a(\partial_j u) + b(x)a(u) = F \text{ in } L^{p'}(G)$$

$$(5.5.b) \quad \sum_{j=1}^n a(\partial_j u) \nu_j + c(s)a(u(s)) = F_0 \text{ in } L^{p'}(\partial G).$$

When $c \equiv 0$ this is a nonlinear *Neumann problem*, a boundary-value problem of *second type*, while for $c > 0$ it is a *Robin problem*, a boundary condition of *third type*.

EXAMPLE 5.C. Problems of *mixed type* (e.g., first and third type) arise as follows. Let Γ_1 be a subset of ∂G and $\Gamma_2 = \partial G \sim \Gamma_1$, the complement. Choose $V = \{u \in W^{1,p}(G) : \gamma u|_{\Gamma_1} = 0\}$. Then the solution to the functional equation (5.4) satisfies the stable boundary condition $u|_{\Gamma_1} = 0$ since $u \in V$, and it also satisfies the third type complementary boundary condition (5.5.b) on Γ_2 , since $Rg\{\gamma|_V\}$ is dense in $L^{p'}(\Gamma_2)$.

EXAMPLE 5.D. Finally, we mention a boundary condition of *non-local* character. For this we choose

$$V = \{ v \in W^{1,p}(G) : \gamma(v) \text{ is constant in } L^p(\partial G) \} .$$

Since the (one-dimensional) subspace of constant functions is closed in $L^p(\partial G)$ and γ is continuous, it follows that V is closed in $W^{1,p}(G)$; as before, $\mathcal{A} : V \rightarrow V'$ is strictly-monotone, continuous, bounded and coercive. If $\mathcal{A}u = f$, as above, then we find

$$(5.6.a) \quad u \in W^{1,p}(G) , \quad - \sum_{j=1}^n \partial_j a(\partial_j u) + b(x)a(u) = F \quad \text{in } L^{p'}(G)$$

$$(5.6.b) \quad \gamma u(s) = c_0 , \quad s \in \partial G , \quad \text{for some } c_0 \in \mathbb{R} , \quad \text{and}$$

$$(5.6.c) \quad \int_{\partial G} \sum_{j=1}^n a(\partial_j u) \nu_j ds + a(c_0) \int_{\partial G} c(s) ds = \int_{\partial G} F_0 \quad \text{in } \mathbb{R} .$$

This is the *Adler problem*, a boundary-value problem of *fourth type*. Note that the constant c_0 is not known a-priori; otherwise, this would be an over-determined Dirichlet problem. Note that (5.6.b) is the stable boundary condition and (5.6.c) is the complementary boundary condition.

In each of the preceding problems we used the same operator $\mathcal{A}$ but varied the space V ; $\mathcal{A} : V \rightarrow V'$ is monotone, continuous and bounded. It is not necessarily coercive: if $b(\cdot)$ and $c(\cdot)$ are both zero, then $\mathcal{A}u(u) = 0$ whenever u is constant. If V contains non-zero constant functions then $\mathcal{A}$ is *not* coercive on V . We shall show these are the *only* ways this $\mathcal{A}$ can fail to be coercive!

We shall use the Rellich-Kandorochov Theorem 4.2 to implement the following general situation.

PROPOSITION 5.2. *Suppose p, q, r are seminorms on a linear space V and that $\|x\| \equiv p(x) + r(x)$, $|x| \equiv p(x) + q(x)$ are norms. Let $B = \{V, \|\cdot\|\}$, $B_1 = \{V, |\cdot|\}$, $B_0 = \{V, r(\cdot)\}$ and assume $B \hookrightarrow B_1$ is continuous, $B \hookrightarrow B_0$ is compact, and B is reflexive. Then $B_1 \hookrightarrow B$ is continuous, hence, $\|\cdot\|$ and $|\cdot|$ are equivalent norms.*

PROOF. Otherwise there is a sequence $\{v_n\}$ such that $|v_n| \rightarrow 0$ and $\|v_n\| = 1$ for $n \geq 1$. Since $\{v_n\}$ is bounded in B , some subsequence (with the same notation) satisfies $v_n \rightarrow v$ in B_0 , $v_n \rightharpoonup v$ in B . But then $v_n \rightarrow v$ in B_1 so $v = 0$. Since $v_n \rightarrow 0$ in B_0 and B_1 , we have $v_n \rightarrow 0$ in B . $\square$

As an immediate application take V a closed subspace of $W^{1,p}(G)$, $1 < p < \infty$, and

$$p(v) = \left(\sum_{j=1}^n \|\partial_j v\|_{L^p(G)}^p \right)^{1/p}, \quad r(v) = \|v\|_{L^p(G)}$$

$$q(v) = \left(\int_G b(x)|v(x)|^p dx \right)^{1/p} + \left(\int_{\partial G} c(s)|v(s)|^p ds \right)^{1/p}, \quad v \in V.$$

where $b \in L^\infty(G)$, $c \in L^\infty(\partial G)$ are non-negative. Then Proposition 5.2 applies if $p + q$ is a norm, that is, if one of $b(\cdot)$ or $c(\cdot)$ is non-identically-zero or if zero is the only constant function in V . Thus, in either case, it follows that $\mathcal{A} : V \rightarrow V'$ is V -coercive and so we obtain existence of solutions.

Finally, we shall extract the essential points of our dissection of $\mathcal{A} : V \rightarrow V'$ into a partial differential equation (5.5.a) and a complementary boundary condition (5.5.b). This construction results in an *abstract Green's theorem*.

Assume V, B are Banach spaces and $\gamma : V \rightarrow B$ is a strict homomorphism with kernel given by $\ker(\gamma) = V_0$. Thus the quotient map is an isomorphism of V/V_0 onto B , and it follows that the dual map $\gamma^*(g) = g \circ \gamma$ defines an isomorphism of B' onto $V_0^\perp \equiv \{f \in V' : f|_{V_0} = 0\}$, the *annihilator* of V_0 in V' .

EXAMPLE 5.E. Take $V = W^{1,p}(G)$, $\gamma = \text{trace}$, and $B \equiv Rg(\gamma)$; it is known that $B = W^{1-1/p,p}(\partial G)$. We have shown that $V_0 = W_0^{1,p}(G)$; since $C_0^\infty(G)$ is dense in V_0 it follows $V_0' \subset \mathcal{D}^*(G)$.

Assume $m_0 : V \times V \rightarrow \mathbb{R}$ is a continuous semi-scalar-product, $|\cdot|$ is a continuous seminorm on V and define $|v|_{\mathcal{X}} \equiv m_0(v, v)^{1/2} + |v|$, $v \in V$; $\mathcal{X} = \{V, |\cdot|_{\mathcal{X}}\}$ is a seminorm space, and $\mathcal{X}' \subset V'$. Assume V_0 is dense in $\mathcal{X}$; then $\mathcal{X}' \subset V_0'$. We call $\mathcal{X}'$ a *pivot space* since its elements belong to both V_0' and V' . Let $M_1 : B \rightarrow B'$ be given.

EXAMPLE 5.E (CONTINUED). Set $m_0(u, v) = \int_G m_0(x)u(x)v(x) dx$ where $m_0(x) \geq 0$ and $m_0 \in L^{p/p-2}(G)$, $p \geq 2$. Then $\mathcal{X}' = \{m_0^{1/2}u : u \in L^2(G)\}$. Similarly construct $M_1\varphi(\psi) = \int_{\partial G} m_1(s)\varphi(s)\psi(s) ds$, $\varphi, \psi \in B$.

Define $\mathcal{M} : V \rightarrow V'$ by $\mathcal{M}u(v) = m_0(u, v) + M_1(\gamma u)(\gamma v)$, for $u, v \in V$, and $M : V \rightarrow V_0'$ by $Mu = \mathcal{M}(u)|_{V_0}$. Thus $Mu \in \mathcal{X}'$ and $Mu(v) = m_0(u, v)$ for $u, v \in V$, since V_0 is dense in $\mathcal{X}$. This gives the identity $\mathcal{M}u(v) = Mu(v) + M_1(\gamma u)(\gamma v)$, $u, v \in V$, hence, $\mathcal{M}(u) = Mu + \gamma' M_1 \gamma(u)$, $u \in V$, a trivial decomposition of this element of V' as the sum of one each from V_0' and from $V_0^\perp$.

Let $\mathcal{A} : V \rightarrow V'$ be given and define the *formal operator* $A(u) = \mathcal{A}(u)|_{V_0}$ to be the indicated restriction and consider it on the domain $D \equiv \{u \in V : \mathcal{A}(u) \in \mathcal{X}'\}$. Then for each $u \in D$, $\mathcal{A}(u) - A(u) \in V_0^\perp$ in V' . Since $Rg(\gamma') = V_0^\perp$, there is a unique $\partial_A u \in B'$ for which $\mathcal{A}(u) - A(u) = \gamma'(\partial_A u)$. This defines the operator $\partial_A : D \rightarrow B'$ for which

$$(5.7) \quad \mathcal{A}u(v) = Au(v) + \partial_A u(\gamma v), \quad u \in D, \quad v \in V.$$

This is the abstract *Green's formula* for $\mathcal{A}$, and ∂_A is the corresponding *boundary operator*. Such formulas will be very useful in a vast variety of situations, and they will be used frequently hereafter. We summarize this elementary construction in the following.

PROPOSITION 5.3. *Assume $\gamma : V \rightarrow B$ is a strict surjective homomorphism between Banach spaces and V_0 is the kernel of γ . Assume $m_0 : V \times V \rightarrow \mathbb{R}$ is a continuous semi-scalar-product and $|\cdot|$ is a continuous seminorm on V ; let $|v|_{\mathcal{X}} = m_0(v, v)^{1/2} + |v|$, and assume V_0 is dense in the seminorm space $\mathcal{X} \equiv \{V, |\cdot|_{\mathcal{X}}\}$. Thus $\mathcal{X}' \subset V'$ and $\mathcal{X}' \hookrightarrow V'_0$. Assume that operators $M_1 : B \rightarrow B'$ and $\mathcal{A} : V \rightarrow V'$ are given; define $M \in \mathcal{L}(V, \mathcal{X}')$ and $\mathcal{M} : V \rightarrow V'$ by $Mu(v) = m_0(u, v)$, $\mathcal{M}u(v) = m_0(u, v) + M_1(\gamma u)(\gamma v)$, $u, v \in V$, so that $\mathcal{M} = M + \gamma' M_1 \gamma$. Define the formal part $A : V \rightarrow V'_0$ of $\mathcal{A}$ by $Au = \mathcal{A}u|_{V'_0}$, $u \in V$, and set $D = \{u \in V : Au \in \mathcal{X}'\}$. Then there is a unique $\partial_A : D \rightarrow B'$ for which (5.7) holds.*

EXAMPLE 5.E (CONTINUED). Let the operator $\mathcal{A}$ on $V = W^{1,p}(G)$ be given by (5.3). Then the formal part is $A(u) = -\sum_{j=1}^n \partial_j a(\partial_j u) + b(x)a(u)$ and D is the set of $u \in W^{1,p}$ for which the partial differential equation takes values in $\mathcal{X}'$, the space determined above by $m_0(x)$. (If $m_0(x) = 1$, then $\mathcal{X}' = L^2(G)$.) Although the boundary operator is defined on all of D , we can compute its value on the smooth functions $u \in D$ by the divergence theorem as

$$\partial_A u(\psi) = \int_{\partial G} \left(\sum_{j=1}^n a(\partial_j u) \nu_j + c(s)a(u(s)) \right) \psi(s) ds, \quad \psi \in Rg(\gamma),$$

just as above. In fact, ∂_A is given on D by the identity (5.7) once we verify that its value depends only on γv .

We show how the preceding construction is useful in representing the solution of an abstract functional equation as a formal part (the partial differential equation) and a complimentary part (the boundary condition in $Rg(\gamma)$); these conditions are in addition to those imposed directly by the space V . Given the spaces and operators above, let $F \in \mathcal{X}'$, $g \in B'$ and define $f \in V'$ by $f = F + \gamma'(g)$. Consider the functional equation

$$(5.8) \quad u \in V : \lambda \mathcal{M}(u) + \mathcal{A}(u) = f \text{ in } V'.$$

That is,

$$\lambda (Mu(v) + M_1(\gamma u)(\gamma v)) + \mathcal{A}u(v) = F(v) + g(\gamma v), \quad v \in V.$$

From our abstract Green's theorem it follows that this is equivalent to

$$(5.9.a) \quad u \in D, \quad \lambda \mathcal{M}(u) + A(u) = F \text{ in } \mathcal{X}', \text{ and}$$

$$(5.9.b) \quad \lambda M_1(\gamma u) + \partial_A(u) = g \text{ in } B'.$$

This is the desired decomposition of the functional equation (5.8) into a partial differential equation (5.9.a) and a complementary boundary condition (5.9.b).

An Elliptic System.

All of our examples above of boundary-value problems for an elliptic equation can be extended in various ways to the case of elliptic systems. However it is often the case that the structure of these systems does *not* permit the direct application of our general existence theorems, especially with respect to monotone or coercive properties. Here we shall describe a model system to which our theory does directly apply. This will be done in the abstract setting developed above. We shall then show by examples that this model system contains a variety of systems of boundary-value problems, including those which are coupled along a common boundary, or

throughout an interior region, or the case of an elliptic problem in the tangent space on a boundary or interior submanifold of a domain which is coupled to a problem on the full domain.

Suppose V_1, V_2, U are reflexive Banach spaces. For $j = 1, 2$, let $\lambda_j \in \mathcal{L}(V_j, U)$ with dual $\lambda'_j \in \mathcal{L}(U', V'_j)$ and assume operators $\mathcal{A}_j : V_j \rightarrow V'_j$ and $\mathcal{B} : U \rightarrow U'$ are given. On the product space $V = V_1 \times V_2$ with elements denoted by $u = [u_1, u_2]$, $u_j \in V_j$, we define $\lambda : V \rightarrow U$ by $\lambda u = \lambda_1 u_1 - \lambda_2 u_2$ and $\mathcal{A} : V \rightarrow V'$ by

$$(5.9) \quad \mathcal{A}u(v) = \mathcal{A}_1 u_1(v_1) + \mathcal{A}_2 u_2(v_2) + \mathcal{B}(\lambda u + u_0)(\lambda v), \quad u, v \in V,$$

where $u_0 \in U$ is given. Thus, $\mathcal{A}u = [\mathcal{A}_1 u_1, \mathcal{A}_2 u_2] + \lambda' \mathcal{B}(\lambda u + u_0)$, $u \in V$, where $\lambda' g = [\lambda'_1 g, -\lambda'_2 g] \in V'$ is the dual of λ . For any $f = [f_1, f_2] \in V'$, the equation $\mathcal{A}u = f$ is equivalent to the system

$$(5.10.a) \quad u_1 \in V_1 : \mathcal{A}_1 u_1 + \lambda'_1 \mathcal{B}(\lambda_1 u_1 - \lambda_2 u_2 + u_0) = f_1 \text{ in } V'_1,$$

$$(5.10.b) \quad u_2 \in V_2 : \mathcal{A}_2 u_2 - \lambda'_2 \mathcal{B}(\lambda_1 u_1 - \lambda_2 u_2 + u_0) = f_2 \text{ in } V'_2.$$

The special form of the coupling λ is motivated by models in which exchange flux or forces are determined by *differences*. However, we need here only that $\lambda \in \mathcal{L}(V, U)$. In order to apply the results of Section 2 to such a system, the following observations are necessary.

PROPOSITION 5.4. *If each of the operators $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}$ is either monotone, hemicontinuous, demicontinuous, continuous, or bounded, then the operator $\mathcal{A}$, as given by (5.9), has the same property. If both of $\mathcal{A}_1, \mathcal{A}_2$ are Type-M and if $\lambda' \mathcal{B} \lambda$ is either completely continuous or monotone and weakly continuous, then $\mathcal{A}$ is Type-M.*

PROOF. The verifications of all but the last property are immediate. The Type-M property follows from Example 2.B and Example 2.C. $\square$

EXAMPLE 5.F. Let G_1 and G_2 be disjoint bounded domains in $\mathbb{R}^n$ which share a common portion Γ of their boundaries: Γ is a manifold of dimension $n - 1$ with $\Gamma \subset \partial G_1 \cap \partial G_2$. Let $U = L^p(\Gamma)$ and $V_j = W^{1,p}(G_j)$ for $j = 1, 2$, and set $\lambda_j = \gamma_j|_\Gamma$, the restriction of the traces to Γ . Choose for simplicity (cf., (5.3))

$$\mathcal{A}_j u(v) = \int_{G_j} \left\{ \sum_{i=1}^n a(\partial_i u) \partial_i v \right\} dx, \quad u, v \in V_j$$

so the formal operators are given by

$$\mathcal{A}_1 u = \mathcal{A}_2 u = - \sum_{j=1}^n \partial_j a(\partial_j u) \quad \text{in } V'_j$$

and the boundary operators are

$$\partial_{\mathcal{A}_j} u = \sum_{i=1}^n a(\partial_i u) \cdot \nu_i^j, \quad j = 1, 2,$$

where $\vec{\nu}^j$ is the unit outward normal on ∂G_j , hence, $\vec{\nu}^1 = -\vec{\nu}^2$ on Γ . Let $F_j \in L^{p'}(G_j)$, $g_j \in L^{p'}(\partial G_j)$ and define

$$f_j(v) = \int_{G_j} F_j v dx + \int_{\partial G_j} g_j \gamma_j v ds, \quad v \in V_j,$$

for $j = 1, 2$. Let $b : \mathbb{R} \rightarrow \mathbb{R}$ be continuous with $|b(\xi)| \leq c(|\xi|^{p-1} + 1)$ and define $\mathcal{B} : U = L^p(\Gamma) \rightarrow U'$ by Nemytskii's Theorem 3.2 as $\mathcal{B}(\psi)(x) = b(\psi(x))$, $x \in \Gamma$. Then the system (5.10) is the generalized formulation of the *boundary-coupled* elliptic system

$$(5.11.a) \quad A_j(u_j) = F_j \text{ in } L^{p'}(G_j), \partial_{A_j} u_j = g_j \text{ in } L^{p'}(\partial G_j \sim \Gamma), \quad j = 1, 2$$

$$(5.11.b) \quad \partial_{A_1} u_1 + b(\gamma_1 u_1 - \gamma_2 u_2) = g_1, \partial_{A_2} u_2 - b(\gamma_1 u_1 - \gamma_2 u_2) = g_2 \text{ in } L^{p'}(\Gamma).$$

Such problems arise as the description of diffusion in two regions G_1, G_2 with a distributed flux $b(\gamma_1 u_1 - \gamma_2 u_2)$ across their common boundary interface Γ , or in elasticity where G_1, G_2 are membranes with a distributed elastic constraint along the interface Γ .

EXAMPLE 5.G. Let $G_0 \subset G$ be bounded domains in $\mathbb{R}^n$ and set $V_j = W_0^{1,p}(G)$, $U = L^p(G_0)$, and let $\lambda_j : V_j \rightarrow U$ be the restriction to G_0 of the imbedding. The dual $\lambda'_j : U' \rightarrow V'_j$ is the zero-extension operator. Define $\mathcal{A}_j, A_j, \partial_{A_j}$ as above on $G_j = G$, $j = 1, 2$. Likewise define f_j as above, but with $g_j = 0$, and construct $\mathcal{B} : L^p(G_0) \rightarrow L^{p'}(G_0)$ from $b(\cdot)$ as above. With this data the system (5.10) corresponds to the *interior-coupled* system.

$$(5.12.a) \quad A_1 u_1 + \lambda'_1 b(u_1 - u_2) = F_1 \text{ in } L^{p'}(G), \quad u_1 = 0 \text{ in } L^p(\partial G),$$

$$(5.12.b) \quad A_2 u_2 - \lambda'_2 b(u_1 - u_2) = F_2 \text{ in } L^{p'}(G), \quad u_2 = 0 \text{ in } L^p(\partial G).$$

Such problems arise as double porosity models of diffusion through a composite media. Each equation results from conservation of fluid within each of the two components of the medium, and the second term corresponds to the flux exchanged between the two component materials. The description of displacements of a parallel pair of membranes which are elastically connected on G_0 is also given in this form.

EXAMPLE 5.H. Let G be a bounded domain in $\mathbb{R}^n$ and let $\Gamma \subset \partial G$ be a smooth manifold of dimension $n-1$. Set $V_1 = W^{1,p}(G)$, $V_2 = W^{1,p}(\Gamma)$, and $U = L^p(\Gamma)$, and define $\lambda_1 = \gamma|_{\Gamma}$, the trace restricted to Γ , $\lambda_2 = I$, the identity. Define $\mathcal{A}_1$ as before on G and construct $\mathcal{A}_2$ similarly with *tangential coordinates* in Γ . (For simplicity, we could assume $\Gamma \subset \mathbb{R}^{n-1}$, i.e., that Γ is *flat*.) Then A_2 is an elliptic operator in the tangential variables and ∂_{A_2} is characterized as a derivative in the direction of the outward normal to $\partial\Gamma$. Choose $\mathcal{B} : L^p(\Gamma) \rightarrow L^{p'}(\Gamma)$ as in Example 5.F. Assume $F_1 \in L^{p'}(G)$, $F_2 \in L^{p'}(\Gamma)$, $g_1 \in L^{p'}(\partial G)$, $g_2 \in L^{p'}(\partial\Gamma)$ are given and define

$$f_1(v) = \int_G F_1 v \, dx + \int_{\partial G} g_1 \gamma v \, ds, \quad v \in V_1,$$

$$f_2(w) = \int_{\Gamma} F_2 w \, ds + \int_{\partial\Gamma} g_2 \gamma_{\Gamma} w \, dt, \quad w \in V_2,$$

where $\gamma_{\Gamma} : W^{1,p}(\Gamma) \rightarrow L^p(\partial\Gamma)$ is the trace operator. A solution of the system (5.10) with such data is characterized by

$$(5.13.a) \quad A_1 u_1 = F_1 \text{ in } L^{p'}(G), \quad \partial_{A_1} u_1 = g_1 \text{ in } L^{p'}(\partial G \sim \Gamma),$$

$$(5.13.b) \quad \partial_{A_1} u_1 + b(\gamma u_1 - u_2) = g_1 \text{ in } L^{p'}(\Gamma),$$

$$(5.13.c) \quad A_2 u_2 - b(\gamma u_1 - u_2) = F_2 \text{ in } L^{p'}(\Gamma), \quad \partial_{A_2} u_2 = g_2 \text{ in } L^{p'}(\partial\Gamma).$$

Here (5.13.a) is a boundary-value problem in the region G and (5.13.c) is a boundary-value problem for an *elliptic equation in the tangent space*. The exchange flux, $b(\gamma u_1 - u_2)$, occurs in the first as a sink on the boundary and in the second as a source in the interior. Such a problem describes diffusion in a region on part of whose boundary is a singular region of such high permeability that there is substantial diffusion tangential to the boundary. Similar problems occur when there is an $(n-1)$ -dimensional submanifold in the interior within which diffusion occurs in the tangential directions. Such models are used to describe flow in narrow fractures in a medium.

EXAMPLE 5.I. Let G be a bounded domain in $\mathbb{R}^n$ for which the hypersurface $\Gamma = \{x = (x', x_n) \in G : x_n = 0\}$ is a domain in $\mathbb{R}^{n-1}$. Set $G_1 = \{x \in G : x_n < 0\}$ and $G_2 = \{x \in G : x_n > 0\}$, and define V_j , λ_j , $\mathcal{A}_j$, f_j for $j = 1, 2$ as in Example 5.F. Choose $U = L^p(\Gamma) \times W_0^{1,p}(\Gamma)$. Let $b_j : \mathbb{R} \rightarrow \mathbb{R}$ be continuous with $|b_j(\xi)| \leq c(|\xi|^{p-1} + 1)$, $j = 1, 2$, and define $\mathcal{B}_1 : L^p(\Gamma) \rightarrow L^{p'}(\Gamma)$ by $\mathcal{B}_1(\varphi)(s) = b_1(\varphi(s))$, $s \in \Gamma$, and $\mathcal{B}_2 : W_0^{1,p}(\Gamma) \rightarrow W_0^{1,p}(\Gamma)'$ by

$$\mathcal{B}_2\varphi(\psi) = \int_{\Gamma} \left(\sum_{j=1}^{n-1} b_2(\partial_j \varphi) \partial_j \psi \right) ds, \quad \varphi, \psi \in W_0^{1,p}(\Gamma).$$

Thus the formal part is given by $B_2\varphi = -\sum_{j=1}^{n-1} \partial_j b_2(\partial_j \varphi)$. We define $\mathcal{B} : U \rightarrow U'$ by

$$\mathcal{B}\varphi(\psi) = \mathcal{B}_1\varphi_1(\psi_1) + \mathcal{B}_2\varphi_2(\psi_2), \quad \varphi = [\varphi_1, \varphi_2], \quad \psi = [\psi_1, \psi_2] \in U.$$

Finally, let $\alpha \in [0, 1]$ and define the space

$$V \equiv \{v = [v_1, v_2] \in V_1 \times V_2 : \alpha\lambda_1 v_1 + (1-\alpha)\lambda_2 v_2 \in W_0^{1,p}(\Gamma)\}.$$

Note that V is a Banach space with the norm

$$\|v\|_V = \|v_1\|_{V_1} + \|v_2\|_{V_2} + \|\alpha\gamma_1 v_1 + (1-\alpha)\gamma_2 v_2\|_{W_0^{1,p}(\Gamma)}.$$

Define $\lambda \in \mathcal{L}(V, U)$ by

$$\lambda v = [\lambda_1 v_1 - \lambda_2 v_2, \alpha\lambda_1 v_1 + (1-\alpha)\lambda_2 v_2], \quad v \in V,$$

and $\mathcal{A} : V \rightarrow V'$ by (5.9). The results of Proposition 5.4 hold for this case, and the equation $\mathcal{A}u = f$ is equivalent to the system

$$\begin{aligned} u_1 \in V_1 : \mathcal{A}_1 u_1 + \lambda_1' \mathcal{B}_1(\lambda_1 u_1 - \lambda_2 u_2) + \alpha \lambda_1' \mathcal{B}_2(w) &= f_1 \text{ in } V_1', \\ w \equiv \alpha \lambda_1 u_1 + (1-\alpha) \lambda_2 u_2 &\in W_0^{1,p}(\Gamma), \\ u_2 \in V_2 : \mathcal{A}_2 u_2 - \lambda_2' \mathcal{B}_1(\lambda_1 u_1 - \lambda_2 u_2) + (1-\alpha) \lambda_2' \mathcal{B}_2(w) &= f_2 \text{ in } V_2'. \end{aligned}$$

From the abstract Green's theorem we find that this system is characterized by

$$(5.14.a) \quad \mathcal{A}_j(u_j) = F_j \text{ in } L^{p'}(G_j), \quad \partial_{A_j} u_j = g_j \text{ in } L^{p'}(\partial G_j \sim \Gamma), \quad j = 1, 2,$$

$$(5.14.b) \quad \partial_{A_1} u_1 + b_1(\gamma_1 u_1 - \gamma_2 u_2) + \alpha B_2(w) = g_1 \text{ and}$$

$$\partial_{A_2} u_2 - b_1(\gamma_1 u_1 - \gamma_2 u_2) + (1-\alpha) B_2(w) = g_2, \text{ in } L^{p'}(\Gamma),$$

$$(5.14.c) \quad w = \alpha \gamma_1 u_1 + (1-\alpha) \gamma_2 u_2, \text{ in } W_0^{1,p}(\Gamma).$$

The pair of equations in $L^{p'}(\Gamma)$ can be written in the equivalent form

$$(5.14.b') \quad \begin{aligned} B_2(w) &= -\partial_{A_1} u_1 - \partial_{A_2} u_2 + g_1 + g_2, \\ (1 - \alpha)[\partial_{A_1} u_1 - g_1 + b_1(\gamma_1 u_1 - \gamma_2 u_2)] &= \alpha[\partial_{A_2} u_2 - g_2 - b_1(\gamma_1 u_1 - \gamma_2 u_2)], \end{aligned}$$

and this alternate form is useful for the applications.

The system (5.14) occurs as a model of diffusion in two regions which are connected by a narrow fissure Γ along their common boundary. Thus, suppose we have two regions G_1, G_2 of different materials and assume the fraction of each material in the fissure varies linearly with respect to the distance from the region. Let $\alpha \in [0, 1]$ denote the location of the interface Γ as a function of the fraction of the width from G_2 , so that $\alpha = 0, 1$ correspond respectively to ∂G_2 and ∂G_1 . We assume the concentration in the fracture varies linearly with the transverse distance, so the concentration w on Γ is given by (5.14.c). The diffusion in the regions G_1, G_2 and on their boundaries away from Γ is described by (5.14.a). The flow in the fissure system is modelled by a transverse component and a tangential flow along Γ . The first equation of (5.14.b') gives the tangential flow field by the total flux coming into the fissure along ∂G_1 and ∂G_2 . According to (5.14.c), the transverse gradient

$$\frac{1}{1 - \alpha}(\gamma_1 u_1 - w) = \frac{1}{\alpha}(w - \gamma_2 u_2) = \gamma_1 u_1 - \gamma_2 u_2$$

is independent of α , $0 < \alpha < 1$. That part of the transverse flux induced in the fissure by this concentration difference is $b_1(\gamma_1 u_1 - \gamma_2 u_2)$; the second equation of (5.14b') fixes the proportion of total flux that enters the fissure from the two sides.

II.6. Variational Inequalities and Quasimonotone Operators

Our objectives here are to illustrate some applications of Theorem 2.3. We begin by constructing a variety of examples of variational inequalities that can be resolved in the form (2.2). Then we show that the class of pseudo-monotone operators contains many quasilinear elliptic equations which are monotone only in their principle part and for which the remaining lower order terms comprise a compact perturbation. This provides a substantial extension of the problems for which existence results can be obtained by the methods of Section 2.

We first describe some examples of variational inequalities for elliptic boundary-value problems. For simplicity we shall take the special class of operators $\mathcal{A} : V \rightarrow V'$ given by (5.3); all of the remarks below apply as well to the more general classes of *monotone* operators as given in Proposition 5.1 and the *quasimonotone* case described below. Thus we consider here

$$(6.1.a) \quad \begin{aligned} \mathcal{A}u(v) &= \int_G \left\{ \sum_{j=1}^N a(\partial_j u(x)) \partial_j v(x) + b(x) a(u(x)) v(x) \right\} dx \\ &\quad + \int_{\partial G} c(s) a(\gamma u(s)) \gamma v(s) ds, \quad u, v \in V, \end{aligned}$$

where V is a closed subspace of $W^{1,p}(G)$, $1 < p < \infty$, G is a bounded domain in $\mathbb{R}^N$, and $a(\xi) = |\xi|^{p-2}\xi$. Let $F \in L^{p'}(G)$, $F_0 \in L^{p'}(\partial G)$ and define $f \in V'$ by

$$(6.1.b) \quad f(v) = \int_G F(x) v(x) dx + \int_{\partial G} F_0(s) \gamma v(s) ds, \quad v \in V.$$

Here $\gamma : V \rightarrow L^p(\partial G)$ is the trace operator, and its range is denoted by $B = \gamma[V]$. For a given closed, convex, non-empty subset K of V we consider the variational inequality (2.2). Note that if $K = V$ this is just the equation $\mathcal{A}(u) = f$ in V' , and if K is a *cone* (i.e., $t(x + y) \in K$ for $x, y \in K$, $t \geq 0$), then (2.2) is equivalent to

$$(6.2) \quad u \in K : \mathcal{A}u(v) \geq f(v), \quad v \in K, \quad \text{and} \quad \mathcal{A}u(u) = f(u).$$

This follows from (2.2) by replacing v with $v + u$ and then setting $v = 0$, and it easily implies (2.2) by subtraction. This characterization of solutions of (2.2) in the case of a cone will be useful below.

EXAMPLE 6.A. BOUNDARY CONSTRAINT. Consider the subset of $L^{p'}(\partial G)$ given by $C \equiv \{\psi \in B : \psi(s) \geq 0, \text{ a.e. } s \in \partial G\}$. This set is closed and convex, and, since γ is continuous and linear, it follows that $K \equiv \{v \in V : \gamma v \in C\}$ is also closed and convex. Furthermore, C and K are cones, so any solution of (2.2) is also a solution of (6.2). To characterize such a solution u , we first set $v = \pm\varphi$ in (6.2), $\varphi \in C_0^\infty(G)$, to obtain

$$(6.3.a) \quad u \in V : \mathcal{A}u = F \text{ in } L^{p'}(G)$$

where $\mathcal{A}$ is the formal part of $\mathcal{A}$ given by

$$\mathcal{A}u = - \sum_{j=1}^N \partial_j a(\partial_j u) + b(x)a(u).$$

Thus we can apply the abstract Green's formula (5.7) to obtain from (6.2)

$$(6.3.b) \quad \gamma u \in B, \langle \partial_A u - F_0, \psi \rangle \geq 0 \text{ for } \psi \in C, \quad \text{and} \quad (\partial_A u - F_0)\gamma u = 0.$$

That is, u is a solution of the elliptic equation (6.3.a) which satisfies in a generalized sense the boundary conditions

$$(6.3.b') \quad \begin{cases} \gamma u \geq 0, \text{ a.e. in } L^p(\partial G), \\ \partial_A u \geq F_0 \text{ a.e. in } L^{p'}(\partial G), \text{ and} \\ (\partial_A u - F_0)\gamma u = 0 \text{ a.e. in } L^1(\partial G). \end{cases}$$

These conditions hold if $\partial_A u \in L^{p'}(\partial G)$, whereas we have in general only $\partial_A u \in B'$. If u is sufficiently smooth then this is the case and we have $\partial_A u$ given by

$$\partial_A u(s) = \sum_{j=1}^n a(\partial_j u)\nu_j + c(s)a(u(s)), \quad \text{a.e. } s \in \partial G,$$

as in Section 5. These pointwise constraints of (6.3.b') are meaningful if we can deduce from regularity theory for solutions of (6.3.a) that $u \in W^{2,p}(G)$. However, (6.3.b) is meaningful without any such additional information on the solution u , and it is equivalent to

$$\gamma u \geq 0 \text{ in } B, \quad \partial_A u - F_0 \geq 0 \text{ in } B', \quad \text{and} \quad \langle \partial_A u - F_0, u \rangle = 0,$$

where the cone C gives the ordering on B and in B' .

The classical form of (6.3) is known as the *Signorini problem* of elasticity. This describes the equilibrium position of an elastic body which is supported at its boundary by a rigid frictionless constraint surface corresponding to $u = 0$. The boundary ∂G is partitioned by the solution into two subregions: Γ_0 , on which $\gamma u = 0$

and $\partial_A u \geq F_0$, and Γ_+ , on which $\gamma u > 0$ and $\partial_A u = F_0$. The functional $\partial_A u - F_0$ is just the force exerted by the constraint on the body; this force is active only on that subregion Γ_0 . If Γ_0 or Γ_+ were known, the solution could be more easily obtained by solving the corresponding mixed Dirichlet-Neumann boundary value problem. But finding this unknown Γ_0 is the essential problem here, analogous to various problems with a “free boundary.”

Problems with constraint (only) in the boundary conditions can be characterized rather elegantly in the abstract situation of Proposition 5.3. Thus, assume we have a generalized trace $\gamma : V \rightarrow B$ with kernel V_0 and range B , a seminorm $|v|_{\mathcal{X}} = m_0(v, v)^{1/2} + |v|$ given by a continuous seminorm and semi-scalar-product on V , so that $\mathcal{X} \equiv \{V, |\cdot|_{\mathcal{X}}\}$ has dual $\mathcal{X}' \subset V'$, and that V_0 is dense in $\mathcal{X}$, so that we can identify the pivot space $\mathcal{X}' \hookrightarrow V'_0$ by restriction. Let $M_1 : B \rightarrow B'$ be given and set

$$\mathcal{M}u(v) = m_0(u, v) + M_1(\gamma u)(\gamma v), \quad u, v \in V$$

to obtain $\mathcal{M} = M + \gamma' M_1 \gamma$, where $M : V \rightarrow V'_0$ is the formal operator obtained from $\mathcal{M}$. Let $\mathcal{A} : V \rightarrow V'$ be given, denote its formal part by Au , $Au = \mathcal{A}u|_{V'_0}$, and then we have the abstract Green's formula

$$\mathcal{A}u(v) = Au(v) + \partial_A u(\gamma v), \quad u \in D, \quad v \in V,$$

where $D = \{u \in V : \mathcal{A}u \in \mathcal{X}'\}$ is the domain of the dual boundary operator $\partial_A : D \rightarrow B'$.

Assume we have $F \in \mathcal{X}'$ and $g \in B'$ given; define $f \in V'$ by $f(v) = F(v) + g(\gamma v)$, $v \in V$. Let C be a closed convex set in B for which $K \equiv \{v \in V : \gamma v \in C\}$ is also non-empty. Consider a solution of the variational inequality

$$(6.4) \quad u \in K : (\lambda \mathcal{M}u + \mathcal{A}u)(v - u) \geq f(v - u), \quad v \in K.$$

Since $K + V_0 = K$ and γ maps K onto C , it follows easily that this is equivalent to

$$(6.4.a) \quad u \in V : \lambda M u + Au = F \text{ in } \mathcal{X}'$$

$$(6.4.b) \quad \gamma u \in C, (\lambda M_1 u + \partial_A u)(\psi - \gamma u) \geq g(\psi - \gamma u), \quad \psi \in C.$$

That is, $u \in D$ and the variational inequality on V' is naturally decoupled into an equation in $\mathcal{X}'$ and a variational inequality on B' .

Finally, we note that the non-homogeneous Dirichlet problem is obtained by setting $C = \{g_0\}$ for a given boundary condition $g_0 \in B$. More generally, we get the mixed Dirichlet-Neumann problem by setting $C = \{\psi \in B : \psi = g_0 \text{ a.e. on } \Gamma_0\}$ for a given $\Gamma_0 \subset \partial G$.

EXAMPLE 6.B. INTERIOR CONSTRAINT. In order to simplify the situation, we choose hereafter $V = W_0^{1,p}(G)$ and set $c = 0$, $F_0 = 0$ in (6.4). Thus we obtain null Dirichlet conditions on ∂G . Define the closed cone

$$K = \{v \in W_0^{1,p}(G) : v(x) \geq 0, \text{ a.e. } x \in G\}$$

and consider (6.2). A solution is characterized by

$$(6.5) \quad u \in W_0^{1,p}(G) : u \geq 0, \quad A(u) - F \geq 0, \quad (A(u) - F)u = 0.$$

The first inequality holds a.e. in G and the second in the dual space V' . The domain G is partitioned by the solution u into two regions

$$G_0 = \{x \in G : u(x) = 0\}, \quad G_+ = \{x \in G : u(x) > 0\}$$

and we have (formally)

$$(6.6) \quad Au = F \text{ in } G_+, \quad \gamma u = 0 \text{ on } \partial G_+, \quad \partial_A u = 0 \text{ on } \partial G_+ \cap \overline{G_0}.$$

If G_+ is known then u can be obtained from the Dirichlet problem implicit in (6.6). However the difficulty is to find such a G_+ for which the third condition in (6.9) holds. Thus the unknown boundary, ∂G_+ , is to be determined from the *pair* of boundary conditions in (6.6).

Such a problem arises in elasticity theory to describe the equilibrium position of a membrane ($N = 2$) which is loaded by a distributed force $F(x)$ and constrained below by an obstacle. According to (6.5) we see the vertical force exerted *upward* by the constraint is $Au - F \geq 0$, and this is non-zero only on G_0 , the set on which the constraint is in contact with the membrane. The essential difficulties in this problem involve the regularity of the solution and of the *free surface* which separates G_0 and G_+ .

EXAMPLE 6.C. GRADIENT CONSTRAINT. With the same space V and operator $\mathcal{A}$ as above, we now consider the closed, convex and *bounded* set

$$K = \{v \in V : \|\vec{\nabla} v(x)\|_{\mathbb{R}^n} \leq 1, \text{ a.e. } x \in G\}.$$

A solution u of (2.2) corresponds formally to the two sets

$$G_0 = \{x \in G : \|\vec{\nabla} u(x)\| < 1\}, \quad G_1 = \{x \in G : \|\vec{\nabla} u(x)\| = 1\}$$

and we have $Au = F$ in G_0 with both of u , $\vec{\nabla} u$ continuous across the “free surface” which separates G_0 and G_1 . This corresponds to the *elastic-plastic torsion problem* in mechanics in which the stress $\vec{\nabla} u$ is pointwise bounded. The domain G is divided into the *elastic region* G_0 where the usual equilibrium equation holds, and a *plastic* region in which the stress is at the threshold at which the material begins to flow. That is, $\|\vec{\nabla} u(x)\| > 1$ in the plastic region corresponds to “yielding” of the material to flow. A solution is clearly Lipschitz continuous. The regularity problem is to show that $\vec{\nabla} u$ is smooth across the free surface.

EXAMPLE 6.D. GLOBAL CONSTRAINT. All of the examples above have involved constraints that are applied pointwise, i.e., locally. Here we consider the closed cone

$$K = \left\{ v \in W_0^{1,p}(G) : \int_G v \, dx \geq 0 \right\}$$

and consider a solution of (6.2). In order to characterize a solution we consider the following.

LEMMA 6.1. *Let $\mathbb{B}$ be a Banach space, $f \in \mathbb{B}'$ and set $C = \{v \in \mathbb{B} : f(v) \geq 0\}$. Then*

$$g \in \mathbb{B}' : g(v) \geq 0, \quad v \in C$$

if and only if $g = cf$ for some $c \geq 0$.

PROOF. If $v \in \ker(f)$, then $\pm v \in C$ and thus $g(v) = 0$. Thus $\ker(f) \subset \ker(g)$ and the result follows since these kernels are hyperplanes in $\mathbb{B}$. $\square$

From Lemma 6.1 we see that a solution of (6.5) is characterized by

$$(6.7) \quad u \in W_0^{1,p}(G) : \int_G u \geq 0, \quad A(u) - F = c \geq 0, \quad c \cdot \int_G u = 0.$$

(This could be shown directly by choosing $v = \partial_j \varphi$, $\varphi \in C_0^\infty(G)$, $1 \leq j \leq n$, since then such a $v \in C$.) Such a problem in elasticity would correspond to the equilibrium displacement u of a membrane ($N = 2$) which is loaded by a distributed force $F(x)$ and constrained below by a minimum volume, such as that of an enclosed incompressible fluid. The *uniform* constraint load c of the fluid on the membrane is non-negative, and it is necessarily zero unless the constraint is attained, i.e., $\int_G u = 0$.

Systems of variational inequalities to which our theory immediately applies can be constructed as in Section 5. Specifically, in the situation of Proposition 5.4, the variational inequality (2.2) is equivalent to a system of the form (5.10). If the constraint set is merely a product, $K = K_1 \times K_2$, with K_j convex in V_j , $j = 1, 2$, this system has the structure of a pair of inequalities on the respective spaces. However, more interesting convex sets in the product are obtained from constraints on the *pair* of components. A good illustration is the following.

LEMMA 6.2. Let $C = \{v = [v_1, v_2] \in \mathbb{R}^2 : v_1 \geq v_2\}$. Then for $u, f \in \mathbb{R}^2$,

$$u \in C : f(v) \geq 0, \quad v \in C$$

is equivalent to

$$u_1 - u_2 \geq 0, \quad f_1 + f_2 = 0, \quad f_1 \geq 0, \quad f_1(u_1 - u_2) = 0.$$

Thus we have either $u_1 = u_2$ and $f_1 = -f_2 \geq 0$, or else $u_1 > u_2$ and $f_1 = f_2 = 0$.

This result follows from the observation that $f(v) = f_1(v_1 - v_2) + (f_1 + f_2)v_2$ with arbitrary $v_2 \in \mathbb{R}$.

If we choose our operators, spaces and data as in (5.11) and set

$$K = \{v \in W^{1,p}(G_1) \times W^{1,p}(G_2) : \gamma_1 v_1 \geq \gamma_2 v_2 \text{ in } L^p(\Gamma)\},$$

then the variational inequality (6.2) is equivalent to

$$(6.8.a) \quad A_j(u_j) = F_j \text{ in } L^{p'}(G_j), \quad \partial_{A_j} u_j = g_j \text{ in } L^{p'}(\partial G_j \sim \Gamma), \quad j = 1, 2,$$

$$(6.8.b) \quad \partial_{A_1} u_1 + \partial_{A_2} u_2 = g_1 + g_2 \text{ in } L^{p'}(\Gamma)$$

$$(6.8.c) \quad \gamma_1 u_1 \geq \gamma_2 u_2, \quad \partial_{A_1} u_1 + b(\gamma_1 u_1 - \gamma_2 u_2) - g_1 \geq 0, \\ \langle \partial_{A_1} u_1 + b(\gamma_1 u_1 - \gamma_2 u_2) - g_2, \gamma_1 u_1 - \gamma_2 u_2 \rangle = 0.$$

Such a problem corresponds to the displacements of a pair of elastic membranes on adjoining regions for which the first is constrained above the second along the common boundary Γ by the force $\partial_{A_1} u_1 + b(\gamma_1 u_1 - \gamma_2 u_2) - g_1$ between them. This force is non-zero only on the subset of Γ where $\gamma_1 u_1 = \gamma_2 u_2$, where they are in contact.

Similarly if we choose our operators, spaces and data as in (5.12) and set

$$K = \{v \in W_0^{1,p}(G) \times W_0^{1,p}(G) : v_1 \geq v_2 \text{ in } L^{p'}(G)\},$$

then (6.2) is formally equivalent to

$$(6.9.a) \quad u_1, u_2 \in W_0^{1,p}(G) : A_1 u_1 + A_2 u_2 = F_1 + F_2 \text{ in } L^{p'}(G) ,$$

$$(6.9.b) \quad u_1 \geq u_2 , \quad A_1 u_1 \lambda'_1 b(u_1 - u_2) \geq F_1 , \\ \langle A_1 u_1 + \lambda'_1 b(u_1 - u_2) - F_1 , u_1 - u_2 \rangle = 0 .$$

Such a problem corresponds to the description of a pair of parallel membranes for which the first is bounded below by the second. The force of the second on the first is $A_1 u_1 + \lambda'_1 b(u_1 - u_2) - F_1$; it is non-negative everywhere and it is non-zero only where $u_1 = u_2$, i.e., where the membranes are in contact.

We next construct a class of nonlinear elliptic operators in divergence form which are monotone only in those terms in the principle part, i.e., the highest order terms. The remaining terms must be of lower order, and this is characterized by a compactness assumption.

Let G be bounded domain in $\mathbb{R}^N$, V a closed subspace of $W^{1,p}(G)$ which contains $W_0^{1,p}(G)$, $1 < p < \infty$, and assume the imbedding $V \hookrightarrow L^p(G)$ is compact. Assume given the functions $a_j : G \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ for $0 \leq j \leq N$ which satisfy

$$(P_1) \quad a_j(x, \eta, \xi) \text{ is measurable in } x \in G \text{ and continuous in } (\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N ,$$

$$(P_2) \quad |a_j(x, \eta, \xi)| \leq c(k(x) + |\eta|^{p-1} + \|\xi\|^{p-1}) , \text{ a.e. } x \in G , \eta \in \mathbb{R} , \xi \in \mathbb{R}^N ,$$

where $k \in L^{p'}(G)$, $1/p + 1/p' = 1$. Define $\mathcal{A} : V \times V \rightarrow V'$ by

$$(6.10) \quad \mathcal{A}(u, v)(w) = \mathcal{A}_1(u, v)(w) + \mathcal{A}_0(u)(w) , \\ \mathcal{A}_1(u, v)(w) = \int_G \left\{ \sum_{j=1}^N a_j(x, u(x), \vec{\nabla} v(x)) \partial_j w(x) \right\} dx , \\ \mathcal{A}_0(u)(w) = \int_G a_0(x, u(x), \vec{\nabla} u(x)) w(x) dx , \quad u, v, w \in V .$$

This operator satisfies the estimate

$$\|\mathcal{A}(u, v)\|_{V'} \leq C \{ \|k\|_{L^{p'}} + \|u\|_V^{p-1} + \|v\|_V^{p-1} \} ,$$

so it follows from Nemytskii's Theorem 3.2 that $\mathcal{A}$ is continuous. Assume also that

$$(P_3) \quad \sum_{j=1}^N (a_j(x, \eta, \xi) - a_j(x, \eta, \tilde{\xi})) (\xi_j - \tilde{\xi}_j) > 0$$

$$\text{for a.e. } x \in G , \eta \in \mathbb{R} , \xi \neq \tilde{\xi} \in \mathbb{R}^N ,$$

$$(P_4) \quad \frac{\sum_{j=1}^N a_j(x, \eta, \xi) \xi_j}{\|\xi\| + \|\xi\|^{p-1}} \rightarrow +\infty \text{ as } \|\xi\| \rightarrow +\infty ,$$

uniformly for η bounded, at a.e. $x \in G$.

From P_3 it follows that $\mathcal{A}(u, v)$ is monotone in v :

$$\langle \mathcal{A}_1(u, v_1) - \mathcal{A}_1(u, v_2), v_1 - v_2 \rangle \geq 0 , \quad u, v_1, v_2 \in V .$$

Our objective is to examine the dependence of $\mathcal{A}(u, u)$ on u with respect to *weak convergence*.

First consider $u_n \rightharpoonup u$ and $w_n \rightharpoonup w$ in V and a fixed $v \in V$. Then we have $u_n \rightarrow u$ and $w_n \rightarrow w$ while $\partial_j u_n \rightharpoonup \partial_j u$ and $\partial_j w_n \rightharpoonup \partial_j w$ in $L^p(G)$. Thus each of $a_j(\cdot, u_n, \vec{\nabla} v) \rightarrow a_j(\cdot, u, \nabla v)$ in $L^{p'}(G)$, so

$$\lim_{n \rightarrow \infty} \mathcal{A}_1(u_n, v)w_n = \mathcal{A}_1(u, v)w .$$

That is, the first part or “higher order” term is harmless.

Consider the second part, $\mathcal{A}_0$. If $w = 0$, it follows that $\mathcal{A}_0(u_n)w_n \rightarrow 0$ since $\{\mathcal{A}_0(u_n)\}$ is bounded in $L^{p'}$. Specifically we have $\lim \mathcal{A}(u_n, v)(u_n - u) = 0$, and this leads to the following result:

if $u_n \rightharpoonup u$ in V , $v \in V$, and $\mathcal{A}(u_n, v) \rightharpoonup f$ in V' , then $\mathcal{A}(u_n, v)(u_n) \rightarrow f(u)$.

Assume further that $\limsup_{n \rightarrow \infty} \mathcal{A}(u_n, u_n)(u_n - u) \leq 0$, as suggested by the definition of pseudomonotone. From above we have $\lim \mathcal{A}(u_n, u)(u_n - u) = 0$ so we obtain

$$\begin{aligned} \limsup \langle \mathcal{A}(u_n, u_n) - \mathcal{A}(u_n, u), u_n - u \rangle &= \\ \lim \langle \mathcal{A}(u_n, u_n) - \mathcal{A}(u_n, u), u_n - u \rangle &= \\ \lim_{n \rightarrow \infty} \int_G F_n(x) dx &= 0 \end{aligned}$$

because the integrands satisfy

$$\begin{aligned} F_n(x) &\equiv \sum_{j=1}^N \left(a_j(x, u_n(x), \vec{\nabla} u_n(x)) - a_j(x, u_n(x), \vec{\nabla} u(x)) \right) (\partial_j u_n(x) - \partial_j u(x)) \\ &\geq 0, \text{ a.e. } x \in G , \end{aligned}$$

by P_3 . Thus $F_n \rightarrow 0$ in $L^1(G)$ and by passing to a subsequence we obtain $F_n(x) \rightarrow 0$ at a.e. $x \in G$. Pick an $x \in G$ for which we have $P_2, P_3, P_4, u_n(x) \rightarrow u(x)$ and $F_n(x) \rightarrow 0$. From P_2 we have

$$F_n(x) \geq \sum_{j=1}^N a_j(x, u_n(x), \nabla u_n(x)) \partial_j u_n(x) - C_x (1 + \|\vec{\nabla} u_n(x)\|_{\mathbb{R}^N}^{p-1} + \|\vec{\nabla} u_n(x)\|_{\mathbb{R}^N})$$

and by P_4 it follows that $\|\vec{\nabla} u_n(x)\|_{\mathbb{R}^N}$ is bounded. Let ξ be an accumulation point of $\{\vec{\nabla} u_n(x)\}$. Then in the limit we obtain

$$\sum_{j=1}^N \left(a_j(x, u(x), \xi) - a_j(x, u(x), \vec{\nabla} u(x)) \right) (\xi_j - \partial_j u(x)) = 0 ,$$

so P_3 shows that $\xi = \vec{\nabla} u(x)$. Applying the preceding argument to an arbitrary subsequence shows that $\vec{\nabla} u_n(x) \rightarrow \vec{\nabla} u(x)$. From P_1 it follows that

$$a_0(x, u_n(x), \vec{\nabla} u_n(x)) \rightarrow a_0(x, u(x), \vec{\nabla} u(x)) , \text{ a.e. } x \in G .$$

Since $\mathcal{A}_0(u_n)$ is bounded in $L^{p'}(G)$, it follows from Proposition 3.4 that $\mathcal{A}_0(u_n) \rightharpoonup \mathcal{A}_0(u)$ in $L^{p'}(G)$. The preceding proves the following.

LEMMA 6.3. Assume $u_n \rightharpoonup u$ and $w_n \rightharpoonup w$ in V , and let $\lim_{n \rightarrow \infty} \sup \mathcal{A}(u_n, u) \cdot (u_n - u) \leq 0$. Then it follows that

$$\lim_{n \rightarrow \infty} \mathcal{A}(u_n, u_n)(u_n - u) = \lim_{n \rightarrow \infty} \mathcal{A}(u_n, u)(u_n - u) = 0$$

and

$$\lim_{n \rightarrow \infty} \mathcal{A}(u_n, v)w_n = \mathcal{A}(u, v)w, \quad v \in V.$$

THEOREM 6.1 (LERAY-LIONS). Let G be a bounded domain in $\mathbb{R}^N$, $1 < p < \infty$, and V a closed subspace, $W_0^{1,p}(G) \subset V \subset W^{1,p}(G)$. Assume that $V \hookrightarrow L^p(G)$ is compact. Let the operator $\mathcal{A} : V \rightarrow V'$ be given by $\mathcal{A}(u) = \mathcal{A}(u, u)$ and (6.13), and assume P_1, P_2, P_3 and P_4 hold. Then $\mathcal{A}$ is pseudo-monotone.

PROOF. Let $u_n \rightharpoonup u$ in V and $\limsup \mathcal{A}(u_n)(u_n - u) \leq 0$. Let $v \in V$, $0 < t < 1$, and set $v_t = (1 - t)u + tv$. Then P_3 gives

$$\langle \mathcal{A}(u_n, u_n) - \mathcal{A}(u_n, v_t), u_n - v_t \rangle \geq 0$$

and writing $u_n - v_t = u_n - u + t(u - v)$ shows that

$$t \langle \mathcal{A}(u_n) - \mathcal{A}(u_n, v_t), u - v \rangle \geq \langle \mathcal{A}(u_n, v_t) - \mathcal{A}(u_n), u_n - u \rangle$$

and the preceding Lemma shows the right side converges to zero as $n \rightarrow \infty$. Take the $\liminf$, divide by $t > 0$, and then let $t \rightarrow 0$ to obtain

$$\liminf \mathcal{A}(u_n)(u_n - v) \geq \mathcal{A}(u)(u - v)$$

as required. $\square$

In the situation of Theorem 6.1, one needs only coercivity to establish the existence of a solution to the corresponding variational inequality or equation. There are various combinations of hypotheses which achieve this; a rather general one is the following.

LEMMA 6.4. Assume P_1, P_2 and

$$(P_5) \quad \begin{aligned} \sum_{j=1}^N a_j(x, \eta, \xi) \xi_j &\geq c_q \|\xi\|^p - K_q (K(x) + |\eta|^q), \\ |a_0(x, \eta, \xi)| &\leq K_q (k(x) + |\eta|^{q-1} + \|\xi\|^{q-1}) \end{aligned}$$

for some q , $1 \leq q < p$, and $K(\cdot) \in L^1(G)$. Let $V \leq \{v \in W^{1,p}(G) : \gamma v(s) = 0, \text{ a.e. } s \in \Gamma_0\}$, where Γ_0 has positive measure in ∂G . Then $\mathcal{A}$ is V -coercive.

PROOF. From P_5 we obtain

$$\mathcal{A}(v, v)(v) \geq c_q \|\vec{\nabla} v\|_{L^p}^p - C \{1 + \|v\|_{L^p} + \|v\|_{L^q}^q + \|\vec{\nabla} v\|_{L^q}^q\}.$$

From Proposition 5.2 it follows that the first term is equivalent to $\|v\|_{W^{1,p}}^p$, and Young's Lemma 3.3.A shows that for each $\varepsilon > 0$ there is a $C_\varepsilon > 1$ for which

$$|s|^q \leq \varepsilon |s|^p + C_\varepsilon, \quad s \in \mathbb{R}.$$

These remarks imply that for some $c_0 > 0$, $C_1 > 1$,

$$\mathcal{A}(v, v)(v) \geq c_0 \|v\|_{W^{1,p}}^p - C_1 (\|v\|_{L^p} + 1), \quad u \in V. \quad \square$$

REMARK. With the Sobolev embedding Theorem 4.3 the similar result can be obtained with $q \leq p + \delta(N)$ for some $\delta(N) > 0$.

Let's characterize the solution of the variational equations for the operator (6.10) as a partial differential equation in G and boundary conditions on ∂G . For this we use the abstract Green's formula (5.7). Thus we choose V to be a subspace of $W^{1,p}(G)$ which contains $V_0 \equiv W_0^{1,p}(G)$ and $B \subset L^p(\partial G)$ to be the range of the trace operator on V , i.e. $\gamma : V \rightarrow B$ is surjective. Choose q , $1 \leq q \leq p$, and $|v|_{\mathcal{X}} = \|v\|_{L^q(G)}$ for $v \in V$. Then $V \hookrightarrow \mathcal{X}$ is continuous and V_0 is dense in $\mathcal{X}$.

Consider $\mathcal{A} : V \rightarrow V'$ given by $\mathcal{A}(u) = \mathcal{A}(u, u)$ as above, where $\mathcal{A}(\cdot, \cdot)$ is defined by (6.10). That is,

$$\mathcal{A}(u)(v) = \int_G \left\{ \sum_{j=1}^N a_j(x, u, \vec{\nabla} u) \partial_j v + a_0(x, u, \vec{\nabla} u) v \right\} dx, \quad u, v \in V.$$

The formal operator, the restriction to V_0 , is given as the generalized function

$$(6.11) \quad A(u) = \mathcal{A}(u)|_{V_0} = - \sum_{j=1}^N \partial_j a_j(\cdot, u, \vec{\nabla} u) + a_0(\cdot, u, \vec{\nabla} u)$$

in V'_0 . The domain of the Green's operator is $D \equiv \{u \in V : A(u) \in L^{q'}(G)\}$, and $\partial_A : D \rightarrow B' \subset L^{p'}(\partial G)$ satisfies

$$Au(u) = Au(v) + \partial_A u(\gamma v), \quad u \in D, \quad v \in V.$$

If we further require that each $a_j(\cdot, u, \vec{\nabla} u) \in W^{1,p}(G)$, ∂G is smooth, and that $u \in W^{2,p}(G)$, respectively, we obtain in succession

$$\begin{aligned} & \int_G \left(\partial_j a_j(x, u(x), \vec{\nabla} u(x)) \cdot v(x) + a_j(x, u(x), \vec{\nabla} u(x)) \cdot \partial_j v(x) \right) dx \\ &= \int_G \partial_j (a_j(\cdot, u, \vec{\nabla} u) \cdot v) dx = \int_{\partial G} \gamma a_j(\cdot, u, \vec{\nabla} u) \nu_j \gamma(v) ds \\ &= \int_{\partial G} a_j(s, \gamma u(s), \gamma(\vec{\nabla} u)) \nu_j(s) \gamma v(s) ds. \end{aligned}$$

Thus the Green's operator is given by

$$(6.12) \quad \partial_A(u)(s) = \sum_{j=1}^N a_j(s, \gamma u(s), \vec{\nabla} u(s)) \nu_j(s)$$

where each term is in $L^{p'}(\partial G)$. Let's choose $V = \{v \in W^{1,p}(G) : \gamma v = 0 \text{ in } L^p(\Gamma_0)\}$ where Γ_0 has positive measure in ∂G . Denote the remainder of the boundary by $\Gamma_1 = \partial G \sim \Gamma_0$. Thus each term of (6.12) is in $L^{p'}(\Gamma_1)$.

Let the data $F \in L^{q'}(G) = \mathcal{X}'$, $g \in L^{p'}(\Gamma_1) \subset B'$ be given and define

$$f(v) = \int_G F(x) v(x) dx + \int_{\Gamma_1} g(s) \gamma v(s) ds, \quad v \in V.$$

Then $f \in V'$ and we have $u \in V$, $\mathcal{A}(u) = f$, if and only if

$$\begin{aligned} & u \in W^{1,p}(G) : A(u) = F \text{ in } L^{q'}(G), \\ & \gamma u = 0 \text{ in } L^p(\Gamma_0), \partial_A u = g \text{ in } L^{p'}(\Gamma_1), \end{aligned}$$

where A and ∂_A are given by (6.14) and (6.15). The existence of such a solution follows from Theorem 2.3 in the special case of $K = V$. Similar results are immediate for variational inequalities by making appropriate choices for K as in the Examples above.

II.7. Convex Functions

For the case of a continuous and symmetric linear operator $\mathcal{A} : V \rightarrow V'$, we saw in Section I.2 that the problem of solving the linear equation $\mathcal{A}u = f$ was equivalent to minimizing the quadratic convex function $\varphi : V \rightarrow \mathbb{R}$ given by $\varphi(u) = \frac{1}{2}\mathcal{A}u(u) - f(u)$. This occurs because $\mathcal{A}$ is the derivative of the first term in φ . Here we shall consider a rather general class of convex functions and find that their derivatives are monotone and hemicontinuous functions (when they exist). More important, the *subgradient* $\partial\varphi$ of a convex function φ extends the notion of derivative to non-smooth functions, and it leads us to the notion of *multi-valued* operators or *relations*. These will be very useful in various contexts. Moreover, we shall extend our existence theorems to include the class of operator equations of the form

$$u \in V : \mathcal{A}(u) + \partial\varphi(u) \ni f \text{ in } V'$$

where $\mathcal{A}$ is pseudo-monotone.

Let V be a Banach space and $\varphi : V \rightarrow \mathbb{R}_\infty \equiv (-\infty, +\infty]$ an extended real-valued function. Then φ is *convex* if

$$\varphi(tu + (1-t)v) \leq t\varphi(u) + (1-t)\varphi(v), \quad u, v \in V, \quad 0 \leq t \leq 1.$$

It is *proper* if $\varphi(u) < \infty$ for some $u \in V$; its *effective domain* is $\text{dom}(\varphi) = \{u \in V : \varphi(u) < \infty\}$. For any set $S \subset V$ we define the *indicator function* by $I_S(u) = 0$ if $u \in S$, $I_S(u) = +\infty$ if $u \notin S$. Then I_S is convex if and only if S is convex, and I_S is proper if and only if S is non-empty.

A problem of general interest is to minimize a convex function φ on a convex set S . This is equivalent to minimizing $\varphi + I_S$ over the whole space V , so it is useful to permit $\varphi(u) = +\infty$. Suppose we had a convex function φ with $\varphi(u) = -\infty$ at some $u \in V$. Then for any $v \in V$ the convexity of φ shows that we can find a $\bar{t} \in [0, 1]$ such that $\varphi = -\infty$ at $tu + (1-t)v$ for $0 \leq t < \bar{t}$ and $\varphi = +\infty$ for $\bar{t} < t \leq 1$. Such functions are too special to be of interest to us, and they unduly complicate our calculations. Hence, we never allow “ $-\infty$ ” as a value for our functions.

The *epigraph* of $\varphi : V \rightarrow \mathbb{R}_\infty$ is given by

$$\text{epi}(\varphi) \equiv \{(u, a) \in V \times \mathbb{R} : \varphi(u) \leq a\}.$$

Note that $u \in \text{dom}(\varphi)$ if and only $(u, a) \in \text{epi}(\varphi)$ for some $a \in \mathbb{R}$. We summarize some elementary properties as follows.

PROPOSITION 7.1. *If φ is convex and $\lambda \geq 0$, then $\lambda\varphi$ is convex. If φ_1 and φ_2 are convex, then $\varphi_1 + \varphi_2$ is convex. If each φ_α is convex, $\alpha \in A$, then $\sup \varphi_\alpha$ is convex and $\text{epi}(\sup_{\alpha \in A} \varphi_\alpha) = \cap \{\text{epi}(\varphi_\alpha) : \alpha \in A\}$. The function $\varphi : V \rightarrow \mathbb{R}_\infty$ is convex, proper, and lower-semi-continuous if and only if $\text{epi}(\varphi)$ is, respectively, convex, non-empty, and closed in $V \times \mathbb{R}$.*

PROOF. The first three statements are elementary. To prove the fourth, assume φ is convex and let $(u, a), (v, b) \in \text{epi}(\varphi)$. Then $\varphi(u) \leq a < \infty$, $\varphi(v) \leq b < \infty$ and for all $t \in [0, 1]$ we have $\varphi(tu + (1-t)v) \leq ta + (1-t)b$, so

$$t(u, a) + (1-t)(v, b) = (tu + (1-t)v, ta + (1-t)b) \in \text{epi}(\varphi).$$

Conversely, if $\text{epi}(\varphi)$ is convex then $\text{dom}(\varphi) = P_V(\text{epi}(\varphi))$ is convex and we need only to consider $u, v \in \text{dom}(\varphi)$. But $\varphi(u) = a < \infty$ and $\varphi(v) = b < \infty$ so from

$$t(u, a) + (1-t)(v, b) = (tu + (1-t)v, ta + (1-t)b) \in \text{epi}(\varphi)$$

we obtain $\varphi(tu + (1-t)v) \leq t\varphi(u) + (1-t)\varphi(v)$. The remaining equivalences are elementary. $\square$

The continuity of a convex function follows from local upper-boundedness.

LEMMA 7.1. *If $\varphi : V \rightarrow \mathbb{R}_\infty$ is convex and upper-bounded on a neighborhood of a point, then φ is continuous at that point.*

PROOF. By translation we obtain an open sphere $S = \{u \in V : \|u\| < r\}$ such that $\varphi(u) \leq 1$ for $u \in S$, $\varphi(0) = 0$. Set $S_\varepsilon = \varepsilon S = \{u \in V : \|u\| < \varepsilon r\}$ with $\varepsilon > 0$. Then for each $u \in S_\varepsilon$ we have $-u/\varepsilon \in S$, so $0 = \varphi(0) = \varphi(u/(1+\varepsilon) + (1-1/(1+\varepsilon))(-u/\varepsilon)) \leq \varphi(u)/(1+\varepsilon) + (1-1/(1+\varepsilon))\varphi(-u/\varepsilon)$, hence, $\varphi(u) \geq -\varepsilon\varphi(-u/\varepsilon) \geq -\varepsilon$. Also, $u/\varepsilon \in S$ implies $\varphi(u) \leq \varepsilon\varphi(u/\varepsilon) + (1-\varepsilon)\varphi(0) \leq \varepsilon$ so we have $|\varphi(u)| \leq \varepsilon$ for all $u \in S_\varepsilon$. $\square$

PROPOSITION 7.2. *If the convex $\varphi : V \rightarrow \mathbb{R}_\infty$ is upper-bounded on a neighborhood of some point, then φ is continuous at each point of the interior of $\text{dom}(\varphi)$.*

PROOF. Let S be as above and $v \in \text{int}(\text{dom}(\varphi))$. Choose $\rho > 1$ such that $\rho v \in \text{dom}(\varphi)$. Then for each $u \in V$ of the form $u = v + (1 - \frac{1}{\rho})w$ with $w \in S$ we have $u = \frac{1}{\rho}(\rho v) + (1 - \frac{1}{\rho})w$ and so $\varphi(u) \leq \frac{1}{\rho}\varphi(\rho v) + (1 - \frac{1}{\rho})$. That is, φ is upper-bounded on a neighborhood of v . $\square$

The proof of Lemma 7.1 shows that φ is actually locally Lipschitz in the $\text{int}(\text{dom}(\varphi))$. Furthermore, it follows that any convex function φ on a finite-dimensional space is continuous on $\text{int}(\text{dom}(\varphi))$. (Take any n -dimensional simplex in $\text{int}(\text{dom}(\varphi))$ and observe that φ is bounded there by its values on the vertices.)

PROPOSITION 7.3. *A proper convex l.s.c. φ on a Banach space V is continuous on $\text{int}(\text{dom}(\varphi))$.*

PROOF. We may suppose $0 \in \text{int}(\text{dom}(\varphi))$ and $\varphi(0) < a$ for some $a \in \mathbb{R}$. The level set $S = \{v \in V : \varphi(v) \leq a\}$ is closed, convex and so also is $B \equiv S \cap (-S)$. The set B is “balanced”: $v \in B$ and $|\lambda| \leq 1$ imply $\varphi(\lambda v) = \varphi(\lambda v + (1-\lambda)0) \leq a$. For any $v \in V$ there is a $\lambda_0 > 0$ such that $|\lambda| \leq \lambda_0$ implies $\lambda v \in \text{int}(\text{dom}(\varphi))$ and by the above remark the restriction of φ to $[-\lambda_0 v, \lambda_0 v]$ is continuous. Thus it is continuous at the origin, so there is a $\lambda_v > 0$ such that $0 \leq \varphi(\lambda v) - \varphi(0) < a - \varphi(0)$ for $|\lambda| \leq \lambda_v$. That is, B is “absorbent”: for $v \in V$ there is a $\lambda_v > 0 : |\lambda| \leq \lambda_v$ implies $\lambda v \in B$. Such a set ... called a *barrel* ... is necessarily a neighborhood. To see this, let B be a barrel. Since B is absorbing, $V = \cup\{nB : n \geq 1\}$. Since V is a complete metric space, it is not of first category, so some nB , hence, B contains

an interior point x_0 . If $x_0 = 0$ we are done. Otherwise, $-x_0 \in \text{int}(B)$, since it is balanced, and by convexity we have $0 = 1/2x_0 + 1/2(-x_0)$ in the interior of B . $\square$

The (*directional derivative* of $\varphi : V \rightarrow \mathbb{R}_\infty$ at $u \in \text{dom}(\varphi)$ in the direction v is the (one-sided) limit

$$\varphi'(u, v) \equiv \lim_{t \downarrow 0} \frac{1}{t} (\varphi(u + tv) - \varphi(u))$$

where it exists. If φ is convex and $u \in \text{dom}(\varphi)$ then for $v \in V$ and $0 < s \leq t$ we have $\varphi(u + sv) = \varphi(\frac{s}{t}(u + tv) + (1 - \frac{s}{t})u) \leq \frac{s}{t}\varphi(u + tv) + (1 - \frac{s}{t})\varphi(u)$, hence,

$$\frac{\varphi(u + sv) - \varphi(u)}{s} \leq \frac{\varphi(u + tv) - \varphi(u)}{t}.$$

Thus the difference-quotient is monotone in t and necessarily has a limit in $[-\infty, +\infty]$. The *G-differential* of $\varphi : V \rightarrow \mathbb{R}_\infty$ at $u \in \text{dom}(\varphi)$ is an $f \in V'$ for which $f(v) = \varphi'(u, v)$ for all $v \in V$. Such an f is unique, it is denoted by $\varphi'(u)$, and we say φ is *G-differentiable* at u .

PROPOSITION 7.4 (KACHUROVSKII). *Let K be convex in V and let $\varphi : V \rightarrow \mathbb{R}_\infty$ be G-differentiable at each $u \in K$, $K = \text{dom}(\varphi)$. The following are equivalent:*

- (a) φ is convex,
- (b) $\varphi'(u)(v - u) \leq \varphi(v) - \varphi(u)$ for all $u, v \in K$, and
- (c) $(\varphi'(u) - \varphi'(v))(u - v) \geq 0$ for all $u, v \in K$.

PROOF. Since $\varphi(u + t(v - u)) \leq t\varphi(v) + (1 - t)\varphi(u)$ it follows that (a) implies (b). By adding to (b) the corresponding inequality for $\varphi'(v)$ it follows that (b) implies (c). To show that (c) implies (a), let $u, v \in K$ and define $g(t) = \varphi(tu + (1 - t)v)$ for $0 \leq t \leq 1$. Then $g'(t) = \varphi'(v + t(u - v))(u - v)$ and for $0 \leq s < t \leq 1$ we have

$$(g'(t) - g'(s))(t - s) = (\varphi'(v + t(u - v)) - \varphi'(v + s(u - v)))(t - s)(u - v) \geq 0$$

so g' is non-decreasing on $[0, 1]$. By the mean-value theorem there follows

$$\frac{g(t) - g(0)}{t - 0} \leq \frac{g(1) - g(t)}{1 - t}, \quad 0 < t < 1,$$

hence, $g(t) \leq tg(1) + (1 - t)g(0)$, and this implies (a). $\square$

Note that since g' is a monotone derivative it is necessarily continuous. Thus the proof shows that the derivative of a convex function is monotone and hemicontinuous.

An *affine function* on V is a function given in the form

$$\ell(v) = c + u^*(v), \quad v \in V$$

where $(u^*, c) \in V' \times \mathbb{R}$. The graph of ℓ is then

$$\text{gr}(\ell) = \{ (v, t) \in V \times \mathbb{R} : \ell(v) = t \} = \{ (v, t) \in V \times \mathbb{R} : -u^*(v) + t = c \},$$

a *hyperplane* in $V \times \mathbb{R}$, given by $h^* = c$ where $h^*(v, t) = -u^*(v) + t = \langle (-u^*, 1)(v, t) \rangle$; it is non-vertical and identified with the *normal* $(-u^*, 1) \in V' \times \mathbb{R}$. The meaning of part (b) of Proposition 7.4 is that the continuous affine function

$$\ell(v) \equiv \varphi(u) + \varphi'(u)(v - u), \quad v \in V$$

is everywhere below $\text{epi}(\varphi)$ and coincides with it at $v = u$. That is, the graph of ℓ is a hyperplane of support to $\text{epi}(\varphi)$ at u . In examples we shall see a correspondence between the smoothness of φ and the uniqueness of such supporting hyperplanes.

DEFINITION. Let $\varphi : V \rightarrow \mathbb{R}_\infty$ be convex and proper. The *subdifferential* of φ at $u \in \text{dom}(\varphi)$ is the set of all functionals $u^* \in V'$ such that

$$u^*(v - u) \leq \varphi(v) - \varphi(u), \quad v \in V,$$

and is denoted by $\partial\varphi(u)$. Each such $u^* \in \partial\varphi(u)$ is also called a subdifferential of φ at u , and when $\partial\varphi(u) \neq \emptyset$ we say φ is *subdifferentiable* at u .

We consider separately the existence and uniqueness of a subdifferential.

PROPOSITION 7.5. *If $\varphi : V \rightarrow \mathbb{R}_\infty$ is convex, proper, and continuous at $u \in V$, then $\partial\varphi(u)$ is closed, convex, bounded, and non-empty.*

PROOF. It is immediate that $\partial\varphi(u)$ is closed and convex, even in general. To show it is bounded, choose $\delta > 0$ so that $\|v - u\| < \delta$ implies $|\varphi(v) - \varphi(u)| \leq 1$. For $u^* \in \partial\varphi(u)$ choose v with $\|v\| = 1$. Then

$$1 \geq \varphi(u + \delta v) - \varphi(u) \geq u^*(\delta v) = \delta u^*(v)$$

so we have $\|u^*\| \leq 1/\delta$.

It remains to show there exists a subgradient at $u \in \text{dom}(\varphi)$. Since φ is continuous at u it follows that $\text{epi}(\varphi)$ has non-empty interior; it contains $S_\delta(u) \times (\varphi(u) + 1, +\infty)$ in $V \times \mathbb{R}$. Thus the convex body $\text{epi}(\varphi)$ lies on one side of a hyperplane containing $(u, \varphi(u))$, a point not in the interior of $\text{epi}(\varphi)$. That is, there exist $u^* \in V'$ and $a \in \mathbb{R}$ such that

$$-u^*(v) + at \geq c \quad \text{for all } (v, t) \in \text{epi}(\varphi)$$

and $-u^*(u) + a\varphi(u) = c$. Now if $a = 0$ we get $u^*(u - v) \geq 0$ for all $v \in \text{dom}(\varphi)$ and since $u \in \text{int}(\text{dom}(\varphi))$ this means $u^* = 0$, a contradiction. Thus we may assume $a = 1$ above, hence,

$$-u^*(v) + t \geq c = -u^*(u) + \varphi(u) \quad \text{for } t \geq \varphi(v),$$

and this shows with $t = \varphi(v)$ that $u^* \in \partial\varphi(u)$. □

PROPOSITION 7.6. *Let $\varphi : V \rightarrow \mathbb{R}_\infty$ be convex and proper. If φ is G -differentiable at $u \in \text{int}(\text{dom}(\varphi))$, then $\partial\varphi(u) = \{\varphi'(u)\}$. If φ is somewhere continuous and $\partial\varphi(u)$ is a singleton, then φ is G -differentiable at u .*

PROOF. Let $u^* \in \partial\varphi(u)$ so $u^*(v - u) \leq \varphi(v) - \varphi(u)$, $v \in V$. Set $v = u + tw$ and let $t \downarrow 0$ to obtain $u^*(w) \leq \varphi'(u)(w)$ for all $w \in V$. Then $u^* = \varphi'(u)$. Since φ is convex we have for each $v \in V$

$$\varphi(u) + t\varphi'(u, v) \leq \varphi(u + tv) \quad \text{for } t \in \mathbb{R},$$

so the affine subset $\{(u + tv, \varphi(u) + t\varphi'(u, v)) : t \in \mathbb{R}\}$ in $V \times \mathbb{R}$ is disjoint from the convex, open and non-empty interior of $\text{epi}(\varphi)$. Thus there is a closed hyperplane containing this affine set and disjoint from $\text{int}(\text{epi}(\varphi))$. This hyperplane is the graph of a continuous affine function below $\text{epi}(\varphi)$ and is exact at $(u, \varphi(u))$. The corresponding functional is the unique subdifferential and agrees with $\varphi'(u, v)$ for all $v \in V$. □

We briefly consider some calculus with subgradients. First note that if $\lambda > 0$ then $\partial(\lambda\varphi) = \lambda\partial\varphi$ for the proper convex φ .

PROPOSITION 7.7. *Let φ_1 and φ_2 be convex functions and suppose there is a point in $\text{dom}(\varphi_1) \cap \text{dom}(\varphi_2)$ at which φ_1 is continuous. Then*

$$\partial(\varphi_1 + \varphi_2) = \partial\varphi_1 + \partial\varphi_2 .$$

PROOF. It is clear that $\partial\varphi_1 + \partial\varphi_2 \subset \partial(\varphi_1 + \varphi_2)$ always holds. Suppose that $u^* \in \partial(\varphi_1 + \varphi_2)(u)$. That is,

$$\varphi_1(v) - \varphi_1(u) - u^*(v - u) \geq \varphi_2(u) - \varphi_2(v) , \quad v \in V ,$$

so the two sets

$$E \equiv \{ (v, t) \in V \times \mathbb{R} : \varphi_1(v) - \varphi_1(u) - u^*(v - u) \leq t \} ,$$

$$F \equiv \{ (v, t) \in V \times \mathbb{R} : \varphi_2(u) - \varphi_2(v) \geq t \}$$

can have only boundary points in common. Moreover, E is convex with non-empty interior, since it is the epigraph of a convex function somewhere continuous. F is the reflection of $\text{epi}(\varphi_2)$, so F is convex. Thus there is a closed hyperplane which separates E and F . It is non-vertical, since otherwise it would separate $\text{dom}(\varphi_1)$ from $\text{dom}(\varphi_2)$, so there exist $u_2^* \in V'$ and $c \in \mathbb{R}$ such that

$$\varphi_1(v) - \varphi_1(u) - u^*(v - u) \geq -u_2^*(v) + c \geq \varphi_2(u) - \varphi_2(v) , \quad v \in V .$$

Setting $v = u$ shows $c = u_2^*(u)$ so we obtain

$$u^* - u_2^* \equiv u_1^* \in \partial\varphi_1(u) , \quad u_2^* \in \partial\varphi_2(u) , \quad u^* = u_1^* + u_2^*$$

as desired. □

Suppose that $\varphi : W \rightarrow \mathbb{R}_\infty$ is convex on the linear space W , that $\Lambda : V \rightarrow W$ is linear, and $\text{dom}(\varphi) \cap \text{Rg}(\Lambda)$ is non-empty. Then the composite $\varphi \circ \Lambda$ is convex and proper on V . For each $w^* \in \partial\varphi(\Lambda u)$ we have

$$w^*(w - \Lambda u) \leq \varphi(w) - \varphi(\Lambda u) , \quad w \in W ,$$

hence, denoting the dual operator by $\Lambda' : W' \rightarrow V'$, we have

$$\Lambda' w^*(v - u) = w^*(\Lambda v - \Lambda u) \leq \varphi \circ \Lambda(v) - \varphi \circ \Lambda(u) , \quad v \in V .$$

That is, $\Lambda' w^* \in \partial(\varphi \circ \Lambda)(u)$ for each $w^* \in \partial\varphi(\Lambda(u))$:

$$\Lambda' \cdot \partial\varphi\Lambda \subset \partial(\varphi \circ \Lambda) .$$

The reverse inclusion and *chain rule* follows by a separation argument.

PROPOSITION 7.8. *Let $\varphi : W \rightarrow \mathbb{R}_\infty$ be convex, $\Lambda : V \rightarrow W$ be continuous and linear, and assume φ is continuous at some point of $\text{Rg}(\Lambda)$. Then $\partial(\varphi \circ \Lambda) = \Lambda' \cdot \partial\varphi \cdot \Lambda$.*

PROOF. (Continued): Let $u^* \in \partial(\varphi \circ \Lambda)(u)$ so

$$u^*(v - u) + \varphi(\Lambda u) \leq \varphi(\Lambda v), \quad v \in V.$$

The affine set $S = \{(\Lambda v, u^*(v - u) + \varphi(\Lambda u)) : v \in V\}$ in $W \times \mathbb{R}$ meets $\text{epi}(\varphi)$ only at boundary points, and $\text{int}(\text{epi} \varphi)$ is non-empty, so there is a non-vertical hyperplane $-w^*(w) + t = c$, $w \in W$ determined by $w^* \in W'$ and $c \in \mathbb{R}$, which is disjoint from $\text{epi}(\varphi)$ and contains S . Thus,

$$-w^*(\Lambda v) + u^*(v - u) + \varphi(\Lambda u) = c, \quad v \in V.$$

Setting $v = u$ shows $c = -w^*(\Lambda u) + \varphi(\Lambda u)$, so

$$w^*(\Lambda(v - u)) = u^*(v - u), \quad v \in V,$$

and we have $u^* = \Lambda'(w^*)$. Finally, this hyperplane is below $\text{epi}(\varphi)$ so

$$w^*(w - \Lambda u) + \varphi(\Lambda u) \leq \varphi(w), \quad w \in W.$$

That is, $w^* \in \partial\varphi(\Lambda u)$ as we desired. $\square$

We show with an elementary example that $Rg(\Lambda) \cap \text{dom}(\varphi) \neq \varnothing$ is not sufficient for the above. Consider $\varphi(s) = -\sqrt{s}$, $s \geq 0$, and $\varphi(s) = +\infty$ for $s < 0$. If $\Lambda = 0$, $W = V = \mathbb{R}$, then $\partial\varphi(0) = \varphi$ but $\partial(\varphi \circ \Lambda(0)) = \{0\}$. The continuity of $\varphi \circ \Lambda$ was used to show the hyperplane is non-vertical, i.e., we use $\text{int} \text{dom}(\varphi \circ \Lambda) \neq \varnothing$. Specifically, suppose that $-w^*(w) + at \geq c \ \forall \ \varphi(w) \leq t$. If $a = 0$ then we obtain $w^*(\Lambda v - \Lambda u) \leq 0 \ \forall \ v \in \text{dom}(\varphi \circ \Lambda)$ and thus $w^* = 0$, a contradiction.

Derivatives arise naturally in minimization problems. Consider a convex set $K \subset V$ and a convex, proper function $\varphi : K \rightarrow \mathbb{R}_\infty$ with $\text{dom}(\varphi) \subset K$. Then the minimization problem

$$u \in K : \varphi(u) \leq \varphi(v), \quad \text{all } v \in K$$

is equivalent to $0 \in \partial\varphi(u)$. Another useful but less obvious criterion for a minimization is given as follows. Let $\tilde{K} = \text{epi}(\varphi) \cap (K \times \mathbb{R}) = \{(x, t) \in V \times \mathbb{R} : x \in K, \varphi(x) \leq t\}$ and $\tilde{\varphi}(x, t) = t$ for $(x, t) \in V \times \mathbb{R}$. Then u minimizes φ on K if and only if

$$u \in K : \varphi(u) \leq t \text{ for all } v \in K \text{ and } t \geq \varphi(v),$$

hence,

$$(u, \varphi(u)) \in \tilde{K} : \tilde{\varphi}(u, \varphi(u)) \leq \tilde{\varphi}(v, t) \text{ for } (v, t) \in \tilde{K}.$$

Note that $\tilde{\varphi}$ is *differentiable*: $\tilde{\varphi}'((u, t)) = (0, 1) \in V' \times \mathbb{R}$.

Let $\mathcal{A} : V \rightarrow V'$ and $f \in V'$. Recall that if K is a closed convex non-empty subset of V and $\varphi = I_K$, the indicator function of K , then the variational inequality

$$(7.1) \quad u \in K : \langle f - \mathcal{A}(u), v - u \rangle \leq 0, \quad v \in K$$

is equivalent to

$$(7.2) \quad u \in V : \langle f - \mathcal{A}(u), v - u \rangle \leq \varphi(v) - \varphi(u), \quad v \in V.$$

That is, (7.1) is equivalent to

$$\mathcal{A}(u) + \partial\varphi(u) \ni f.$$

Consider a general convex, proper function φ and let $\tilde{V} = V \times \mathbb{R}$, $\tilde{K} = \text{epi}(\varphi)$, $\tilde{f} = (f, 0) \in \tilde{V}'$ and $\tilde{\mathcal{A}}(v, t) = (\mathcal{A}v, 1) \in \tilde{V}'$ for $(u, t) \in \tilde{V}$. Then $\langle \tilde{f} - \tilde{\mathcal{A}}\tilde{u}, \tilde{v} - \tilde{u} \rangle = \langle f - \mathcal{A}(u), v - u \rangle - (t - s)$ for

$$\tilde{u} = (u, s), \quad \tilde{v} = (v, t) \in \tilde{V}.$$

If u is a solution of (7.2) then $\tilde{u} \equiv (u, \varphi(u)) \in \tilde{K}$ and

$$\langle \tilde{f} - \tilde{\mathcal{A}}(\tilde{u}), \tilde{v} - \tilde{u} \rangle = \langle f - \mathcal{A}(u), v - u \rangle - (t - \varphi(u)) \leq 0$$

whenever $t \geq \varphi(v)$, $v \in V$, so

$$(7.3) \quad \tilde{u} \in \tilde{K} : \langle \tilde{f} - \tilde{\mathcal{A}}(\tilde{u}), \tilde{v} - \tilde{u} \rangle \leq 0, \quad \tilde{v} \in \tilde{K}.$$

Conversely, if $\tilde{u} = (u, s)$ is a solution of (7.3), then for all $v \in V$, $t \geq \varphi(v)$

$$\langle f - \mathcal{A}(u), v - u \rangle \leq t - s \leq t - \varphi(u)$$

so (7.2) follows. These remarks prove the following.

LEMMA 7.2. *u is a solution of (7.2) if and only if $(u, \varphi(u))$ is a solution of (7.3).*

By combining Lemma 7.2 with Theorem 2.3 we obtain the following.

THEOREM 7.1. *Let V be a separable reflexive Banach space, $\mathcal{A} : V \rightarrow V'$ be bounded and pseudo-monotone, and $f \in V'$. Let $\varphi : V \rightarrow \mathbb{R}_\infty$ be proper, convex, lower semi-continuous, and assume there is a $v_0 \in \text{dom}(\varphi)$ and $R > 0$ such that*

$$(7.4) \quad \langle \mathcal{A}v - f, v - v_0 \rangle + \varphi(v) - \varphi(v_0) > 0 \quad \text{for } \|v\| \geq R.$$

Then there exists a solution of

$$u \in V : \langle f - \mathcal{A}(u), v - u \rangle \leq \varphi(v) - \varphi(u), \quad v \in V.$$

PROOF. We need only to check that the coercivity condition (7.4) implies the corresponding estimate in Theorem 2.3. Note that with $\tilde{v}_0 \equiv (v_0, \varphi(v_0))$ we have for each $\tilde{v} = (v, t) \in K \equiv \text{epi}(\varphi)$

$$\begin{aligned} \langle \tilde{\mathcal{A}}\tilde{v} - \tilde{f}, \tilde{v} - \tilde{v}_0 \rangle &= \langle \mathcal{A}v - f, v - v_0 \rangle + t - \varphi(v_0) \\ &\geq \langle \mathcal{A}v - f, v - v_0 \rangle + \varphi(v) - \varphi(v_0). \end{aligned}$$

If $\langle \tilde{\mathcal{A}}(\tilde{v}) - \tilde{f}, \tilde{v} - \tilde{v}_0 \rangle \leq 0$, then $(*)$ implies $\|v\| < R$, so

$$\begin{aligned} t &\leq \sup_{\|w\| \leq R} \langle f - \mathcal{A}(w), w - v_0 \rangle + \varphi(v_0) \quad \text{and} \\ t &\geq \inf \{ \varphi(w) : \|w\| \leq R \}, \end{aligned}$$

i.e., $|t| \leq C(R)$. That is, $\langle \tilde{\mathcal{A}}(\tilde{v}) - \tilde{f}, \tilde{v} - \tilde{v}_0 \rangle \leq 0$ implies $\|\tilde{v}\|^2 < R^2 + C(R)^2 \equiv \rho^2$ as desired. $\square$

COROLLARY 7.1. *If there is a $v_0 \in \text{dom}(\varphi)$ such that*

$$\lim_{\|v\| \rightarrow \infty} \left(\frac{\langle \mathcal{A}v - f, v - v_0 \rangle + \varphi(v)}{\|v\|} \right) = +\infty,$$

then $Rg(\mathcal{A} + \partial\varphi) = V'$.

II.8. Examples

EXAMPLE 8.A. CONVEX FUNCTIONS ON $\mathbb{R}$.

Let $F : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ be a monotone function. It is then continuous at all but a countable number of points. Set $F^-(x) = \lim_{y \rightarrow x^-} F(y)$, $F^+(x) = \lim_{y \rightarrow x^+} F(y)$, so $F^- = F^+$ a.e. on $\text{dom}(F)$. We then define

$$(8.1) \quad \varphi(x) = \int_{x_0}^x F^-(s) ds = \int_{x_0}^x F^+(s) ds ,$$

where $x_0 \in \text{dom}(F) \subset \text{dom}(\varphi)$ is given. Note that φ is convex and there is an interval $(a, b) \subset \text{dom}(F) \subset \text{dom}(\varphi) \subset [a, b]$. The subgradient is characterized by

$$y \in \partial\varphi(x) \text{ if and only if } y(\xi - x) \leq \int_x^\xi F^- \text{ for } \xi \in \mathbb{R} .$$

To see this, note first that if $y \leq F^+(x)$ then $x \leq \xi$ implies $y(\xi - x) \leq \int_x^\xi F^+$, and if $y \geq F^-(x)$ then $\xi \leq x \Rightarrow \int_\xi^x F^- \leq y(x - \xi)$ which implies the above. Hence, $y \in [F^-(x), F^+(x)]$ implies $y \in \partial\varphi(x)$. The converse follows, since F^- is left-continuous and F^+ is right-continuous, so

$$\partial\varphi(x) = [F^-(x), F^+(x)] , \quad x \in \mathbb{R} .$$

EXAMPLE 8.B. CONVEX INTEGRANDS.

PROPOSITION 8.1. *Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}_\infty$ be proper, convex, lower semi-continuous and either $0 = \varphi(0) = \min(\varphi)$ or the measurable $\Omega \subset \mathbb{R}^n$ has finite measure. Define $\Phi : L^p(\Omega) \rightarrow \mathbb{R}_\infty$, $1 \leq p < \infty$, by*

$$(8.2) \quad \Phi(u) = \int_\Omega \varphi(u(x)) dx \text{ if } \varphi(u) \in L^1(\Omega) , \quad +\infty \text{ otherwise.}$$

Then Φ is proper, convex, lower semi-continuous, and $f \in \partial\Phi(u)$ if and only if

$$f \in L^{p'}(\Omega) , u \in L^p(\Omega) \text{ and } f(x) \in \partial\varphi(u(x)) , \text{ a.e. } x \in \Omega .$$

PROOF. Each set $\{y : \varphi(y) > a\}$ is open and u is measurable, so $\{x : \varphi(u(x)) > a\}$ is measurable for each $a \in \mathbb{R}$, hence $\varphi \circ u$ is measurable. Let's show Φ is LSC. Since φ has an affine lower bound, we may assume it is non-negative. (Otherwise, we add that lower bound to obtain a non-negative φ .) If $u_n \rightarrow u$ in L^p and $\Phi(u_n) \leq r$ for $n \geq 1$, there is a subsequence for which (after a change of notation) we have $u_n(x) \rightarrow u(x)$ a.e. $x \in \Omega$. Since φ is LSC, $\varphi(u(x)) \leq \lim_{n \rightarrow \infty} \inf \varphi(u_n(x))$, a.e. x . Fatou's Lemma shows $\Phi(u) \leq \lim_{n \rightarrow \infty} \inf \Phi(u_n) \leq r$.

Suppose $f \in L^{p'}$, $u \in L^p$ and $f(x) \in \partial\varphi(u(x))$, a.e. x . Then $f(x)(v(x) - u(x)) \leq \varphi(v(x)) - \varphi(u(x))$ and $\varphi \circ u \in L^1$; an integration shows $f \in \partial\Phi(u)$. Conversely, let

$$\int_\Omega f(x)(v(x) - u(x)) dx \leq \int_\Omega (\varphi(v(x)) - \varphi(u(x))) dx , \quad v \in L^p .$$

For each measurable $M \subset \Omega$ we set $w(x) = v(x)$ if $x \in M$ and $w(x) = u(x)$ if $x \in \Omega \setminus M$. Then $w \in L^p$ and

$$\int_M \left\{ f(x)(v(x) - u(x)) - \varphi(v(x)) + \varphi(u(x)) \right\} \leq 0$$

for each such M , so we have

$$f(x)(v(x) - u(x)) \leq \varphi(v(x)) - \varphi(u(x)) , \quad \text{a.e. } x \in \Omega ,$$

for each $v \in L^p$, so $f(x) \in \partial\varphi(u(x))$, a.e. x . $\square$

EXAMPLE 8.C. BOUNDARY FUNCTIONALS.

Let $1 < p < \infty$, G be a bounded domain in $\mathbb{R}^n$, the boundary ∂G be a C^1 manifold of dimension $n-1$, and denote by γ the trace operator from $W^{1,p}(G)$ into $L^p(\partial G)$. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be convex and continuous and suppose it satisfies

$$|\varphi(s)| \leq C(|s|^p + 1) , \quad s \in \mathbb{R} .$$

Thus we define

$$(8.3) \quad \Phi(u) = \int_{\partial G} \varphi(\gamma u(s)) ds , \quad u \in W^{1,p}(G) .$$

Since Φ is a composite of a convex and a linear function as in Proposition 7.8, the function Φ is convex and continuous, and its subgradient is given by

$$\partial\Phi(u) = \gamma'(\partial\varphi(\gamma u)) , \quad u \in W^{1,p} ,$$

where the dual $\gamma' : L^{p'}(\partial G) \rightarrow (W^{1,p})'$ is given by

$$\gamma'g(v) = \int_{\partial G} g(s)\gamma v(s) ds , \quad v \in W^{1,p} ,$$

for $g \in L^{p'}(\partial G)$. That is, $F \in \partial\Phi(u)$ if and only if there is a $g \in \partial\varphi(\gamma u)$ in $L^{p'}(\partial G)$ for which

$$F(v) = \int_{\partial G} g(s)\gamma v(s) ds , \quad v \in W^{1,p} .$$

EXAMPLE 8.D. DIRICHLET INTEGRANDS.

For each integer k , $0 \leq k \leq n$, let there be given a continuous, convex function $\varphi_k : \mathbb{R} \rightarrow \mathbb{R}$ which satisfies

$$\varphi_k(s) \leq C(|s|^p + 1) , \quad s \in \mathbb{R} ,$$

for some p , $1 < p < \infty$. Define

$$(8.4) \quad \Phi(u) = \sum_{k=0}^n \int_G \varphi_k(\partial_k u(x)) dx , \quad u \in W^{1,p}(G)$$

where G is a domain in $\mathbb{R}^n$. Note that Φ is a sum of continuous, convex functions on the Sobolev space $W^{1,p}(G)$, each of which is the composition of a convex integrand on $L^p(G)$ following the continuous linear $\partial_k : W^{1,p} \rightarrow L^p$. By Propositions 7.7 and 7.8, the subgradient can be computed term-by-term and is given formally by

$$\partial\Phi(u) = \sum_{k=0}^n \partial'_k(\partial\varphi_k(\partial_k u)) , \quad u \in W^{1,p}(G) ,$$

where the dual $\partial'_k : L^{p'} \rightarrow (W^{1,p})'$ is given by

$$\partial'_k f(v) = \int_G f(x) \partial_k v(x) dx , \quad v \in W^{1,p}(G) ,$$

for $1 \leq k \leq n$ and $f \in L^{p'}$; $\partial'_0 = \partial_0$ is the identity. To be precise, we have $F \in \partial\Phi(u)$ if and only if there exists $f_k \in \partial\varphi_k(\partial_k u)$ in $L^{p'}$ for each k , $0 \leq k \leq n$, for which

$$F(v) = \sum_{k=0}^n \int_G f_k(x) \partial_k v(x) dx, \quad v \in W^{1,p}(G).$$

By restricting this functional to $V_0 = W_0^{1,p}(G)$ we see the *formal part* is the distribution

$$F_0 = - \sum_{k=1}^n \partial_k f_k + f_0 \in V'_0$$

and we denote this by

$$(\partial\Phi)_0(u) = - \sum_{k=1}^n \partial_k \partial\varphi_k(\partial_k u) + \partial\varphi_0(u).$$

Note that by the classical Green's Theorem if the boundary of G is smooth and each $\partial_k f_k \in L^{p'}$, $1 \leq k \leq n$, then

$$F(v) - F_0(v) = \int_{\partial G} \left\{ \sum_{k=1}^n f_k(s) \nu_k(s) \right\} v(s) ds, \quad v \in V.$$

As in Section 5 we construct an *abstract Green's Theorem* for $\partial\Phi$ for which

$$(8.5) \quad \partial\Phi(u) = (\partial\Phi)_0(u) + \gamma'(\partial_\Phi(u)), \quad u \in D,$$

where $D = \{u \in W^{1,p}(G) : (\partial\Phi)_0(u) \cap L^{p'}(G) \neq \emptyset\}$ and $\partial_\Phi : D \rightarrow B'$ is the corresponding *boundary operator* onto the dual of $B = Rg(\gamma)$. Finally we remark that it is the sum of a Dirichlet integrand and a boundary functional that frequently arises in applications.

Let's consider again the general problem of minimizing the proper, convex $\varphi : V \rightarrow \mathbb{R}_\infty$. Note that u is a solution of

$$u \in V : \varphi(u) \leq \varphi(v), \quad v \in V$$

if and only if $0 \in \partial\varphi(u)$. Also, if K is a convex subset of V which contains a $v_0 \in \text{dom}(\varphi)$, then u is a solution of

$$u \in K : \varphi(u) \leq \varphi(v), \quad v \in K$$

if and only if u minimizes $\varphi + I_K$ over V , and this is equivalent to $0 \in \partial(\varphi + I_K)(u)$. In addition, if φ is continuous at some $v_0 \in K$ then u minimizes φ on K if and only if $0 \in \partial\varphi(u) + \partial I_K(u)$ by Proposition 7.7, and this is equivalent to

$$u \in K, \quad f \in \partial\varphi(u), \quad f(v - u) \geq 0, \quad \text{all } v \in K.$$

Thus, one obtains a characterization of solutions of variational inequalities for multi-valued operators (subgradients) as minima of convex functions on the whole space or as minima on the convex set. The following is an easy sufficient condition for the existence of minimizers.

THEOREM 8.1 (WEIERSTRASS). *If K is convex and closed in a reflexive Banach space V , $\varphi : V \rightarrow \mathbb{R}_\infty$ is lower-semi-continuous and convex with $\text{dom}(\varphi) \cap K \neq \emptyset$, and $\varphi(v) \rightarrow \infty$ as $\|v\| \rightarrow \infty$ with $v \in K$, then φ attains its infimum on K .*

In most applications the convex functional is identified as the *energy* associated with the state u of a system. As in the preceding examples this is frequently a *local* function of u , i.e., it depends on the values of u or its derivatives at each point in the domain G . Our next example is non-local: the function depends on the “total energy” or “total flux” of the system.

PROPOSITION 8.2. *Let A, B be continuous symmetric monotone linear operators from V to V' and $a, b \in \mathbb{R}$. The function*

$$\varphi(u) = 1/2 \max \{ Au(u) + a, Bu(u) + b \}, \quad u \in V,$$

is convex and continuous. Its subgradient is given by

$$\partial\varphi(u) = \begin{cases} \{A(u)\} & \text{if } Au(u) + a > Bu(u) + b, \\ \{\lambda A(u) + (1 - \lambda)Bu, 0 \leq \lambda \leq 1\} & \text{if } Au(u) + a = Bu(u) + b, \\ \{B(u)\} & \text{if } Au(u) + a < Bu(u) + b. \end{cases}$$

PROOF. We need only to verify the computations of the subgradient, and the first and last cases follow from G -differentiability of φ in the respective regions. In the middle case we obtain

$$t^{-1}(\varphi(u + tv) - \varphi(u)) = \max \left\{ Au(v) + \frac{t}{2} Av(v), Bu(v) + \frac{t}{2} Bv(v) \right\},$$

so we have the equivalence of $f \in \partial\varphi(u)$,

$$(8.6) \quad f(v) \leq t^{-1}(\varphi(u + tv) - \varphi(u)), \quad v \in V, \quad t > 0,$$

and of

$$(8.6') \quad f(v) \leq \max \{ Au(v), Bu(v) \}, \quad v \in V.$$

This last condition is equivalent to $f = \lambda Au + (1 - \lambda)Bu$ for some $\lambda, 0 \leq \lambda \leq 1$. For this we use the following.

LEMMA 8.1. *Let $f, f_1, f_2 \in V^*$. Then f is a linear combination of f_1 and f_2 if and only if the $\ker(f) \supset \ker(f_1) \cap \ker(f_2)$.*

PROOF. The “only if” is clear. Suppose $\ker(f) \supset \ker(f_1)$. If f_1 is not identically zero, $f_1(v_0) \neq 0$, then $f(v) = (f(v_0)/f_1(v_0))f_1(v)$, since $\ker(f_1)$ has codimension 1. That is, f is a multiple of f_1 . Suppose $\ker(f) \supset \ker(f_1) \cap \ker(f_2)$. For each $v \in \ker(f_2)$ we have $f(v) = 0$ whenever $f_1(v) = 0$. Thus there is an $a_1 \in \mathbb{R} : f(v) = a_1 f_1(v)$ for $v \in \ker(f_2)$. That is $f - a_1 f_1$ vanishes on $\ker(f_2)$ so as above $f - a_1 f_1 = a_2 f_2$ for some $a_2 \in \mathbb{R}$. $\square$

To finish Proposition 8.2, we note that the condition (8.6') on f implies

$$\min \{ Au(v), Bu(v) \} \leq a_1 Au(v) + a_1 Bu(v) \leq \max \{ Au(v), Bu(v) \}$$

where $f_1 = Au, f_2 = Bu$. $\square$

EXAMPLE 8.E. ENERGY-DEPENDENT ELLIPTIC EQUATIONS.

Choose $V = W_0^{1,2}(G)$, $f(v) = \int_G Fv$, $v \in V$, where F is given in $L^2(G)$ and define

$$(8.7) \quad \varphi(v) = \frac{1}{2} \int_G |\vec{\nabla} v|^2 + \frac{1}{2} \max \left\{ 1, \int_G |v|^2 \right\}, \quad v \in V.$$

Then $\partial\varphi(u) \ni f$ is equivalent to

$$u \in W_0^{1,2}(G) : -\Delta u + \operatorname{sgn}^+ \left(\int_G |u|^2 dx - 1 \right) u \ni F.$$

EXAMPLE 8.F. FLUX-DEPENDENT EQUATION.

Define V and f as before but set

$$(8.8) \quad \varphi(v) = \frac{1}{2} \max \left\{ 1, \int_G |\vec{\nabla} v|^2 \right\}, \quad v \in V.$$

Then $\partial\varphi(u) \ni f$ is characterized by

$$u \in W_0^{1,2}(G) : -\operatorname{sgn}^+ \left(\int_G |\nabla u|^2 - 1 \right) \Delta u \ni F.$$

In these examples we have used the real-valued subgradient

$$\operatorname{sgn}^+(r) = \begin{cases} \{0\}, & \text{if } r < 0, \\ [0, 1], & \text{if } r = 0, \\ \{1\}, & \text{if } r > 0. \end{cases}$$

This is the *positive sign* or *Heaviside relation*. Also see Proposition 8.6 below.

We consider again the relationship between subgradients and the directional derivative,

$$\varphi'(u, v) = \lim_{t \rightarrow 0^+} t^{-1} (\varphi(u + tv) - \varphi(u)).$$

This will be used to study the differentiability of the norm and compositions with it. Thus let V be a normed linear space and assume $\varphi : V \rightarrow \mathbb{R}_\infty$ is proper and convex.

LEMMA 8.2. *The following are equivalent for an $f \in V'$:*

- (a) $f \in \partial\varphi(u)$, i.e., $f(v - u) \leq \varphi(v) - \varphi(u)$, $v \in V$,
- (b) $f(v - u) \leq \varphi'(u, v - u)$, $v \in V$, and
- (c) $f(v - u) \leq t^{-1}(\varphi(u + t(v - u)) - \varphi(u))$, $0 < t \leq 1$, $v \in V$.

PROOF. If we set $v = u + t(w - u)$ in (a) we obtain

$$f(w - u) \leq t^{-1} (\varphi(u + t(w - u)) - \varphi(u)) \leq \varphi(w) - \varphi(u), \quad 0 < t \leq 1.$$

Recall that the difference quotient is monotone in t , so the limit as $t \rightarrow 0^+$ exists in $\mathbb{R}_\infty$. $\square$

For the special case of the norm, $\varphi(v) = \|v\|$, on V we obtain yet another equivalent statement,

- (d) $\|f\| = 1$ and $f(u) = \|u\|$.

This equivalence is straightforward and will be obtained more generally below, but first we characterize the G -differentiability of the norm. Note by Proposition 7.6 this occurs exactly when the subgradient is a singleton.

DEFINITION. A normed space is *strictly convex* if $u \neq v$, $\|u\| = \|v\| = 1$ and $0 < t < 1$ imply $\|tu + (1-t)v\| < 1$.

PROPOSITION 8.3. *The following are equivalent:*

- (a) V is strictly convex.
- (b) $u \neq v$, $\|u\| = \|v\| = 1$ imply $\|tu + (1-t)v\| < 1$ for some $t \in (0, 1)$.
- (c) Any convex subset of the unit sphere contains at most one point.
- (d) Any non-zero $f \in V'$ takes its maximum value on the unit sphere at most once.

PROOF. That (b) $\Rightarrow$ (a) follows by an easy convexity argument; we check that (a) $\Rightarrow$ (c) $\Rightarrow$ (b), so the first three are equivalent. If $\|u\| = \|v\| = 1$ and $f(u) = f(v) = \|f\| \neq 0$, then for $t \in (0, 1)$

$$\|f\| = f(tu + (1-t)v) \leq \|f\| \|tu + (1-t)v\|$$

so $\|tu + (1-t)v\| = 1$ and $u = v$ by (a). Thus (a) $\Rightarrow$ (d). Conversely, if $\|u\| = \|v\| = \|1/2(u+v)\| = 1$, there is an $f \in V' : \|f\| = 1$ and $f(u+v) = 2$. Since $f(u) \leq 1$ and $f(v) \leq 1$ we have $f(u) = f(v)$ so $u = v$ by (d). $\square$

COROLLARY 8.1. *The norm on V is G -differentiable at each $u \neq 0$ if and only if the dual space V' is strictly convex.*

PROOF. Each non-zero $u \in V \subset V''$ attains its maximum on the unit sphere of V' at most once by (d) of Lemma 2 and of Proposition 8.3. $\square$

The next two results will be useful for obtaining estimates on solutions of evolution equations.

PROPOSITION 8.4. *Let $u, v \in V$. Then $\|u + tv\| \geq \|u\|$ for all $t > 0$ if and only if there is an $f \in \partial\varphi(u)$ such that $f(v) \geq 0$, where $\varphi(w) = \|w\|$ above.*

PROOF. Let f be as given. Then $\|f\| \|u\| \leq f(u + tv) \leq \|f\| \|u + tv\|$ so the result follows if $\|f\| \neq 0$, and it is trivial if $\|f\| = 0$. Conversely, if $\|u\| \leq \|u + tv\|$ for $t > 0$ we select for each $t > 0$ an $f_t \in \partial\varphi(u + tv)$. Then

$$\|u\| \leq \|u + tv\| = f_t(u + tv) = f_t(u) + tf_t(v) \leq \|u\| + tf_t(v) \leq \|u\| + t\|v\| ,$$

so we have $f_t(v) \geq 0$, $t > 0$, and $\lim_{t \downarrow 0} f_t(u) = \|u\|$. Since the unit sphere of V' is w^* -compact, there is a subsequence $\{f_{t_n}\}$ w^* -convergent to an $f \in V'$. Then $\|f\| \leq 1$ and $f(u) = \|u\|$, so $\|f\| = 1$ and $f \in \partial\varphi(u)$ with $f(u) \ni 0$. $\square$

The derivative of the norm of a function is computed as follows.

PROPOSITION 8.5. *If the function $u : [0, T) \rightarrow V$ is right-differentiable at $t \in [0, T)$, i.e.,*

$$u^+(t) = \lim_{h \rightarrow 0^+} h^{-1}(u(t+h) - u(t)) \text{ in } V ,$$

then so also is $\varphi(u(t)) \equiv \|u(t)\|$ and

$$\frac{d^+}{dt} \|u(t)\| = \varphi'(u(t), u^+(t)) .$$

PROOF. For sufficiently small $h > 0$ we have

$$\begin{aligned} & |(\|u(t+h)\| - \|u(t)\|) - (\|u(t) + hu^+(t)\| - \|u(t)\|)| \\ &= | \|u(t+h)\| - \|u(t) + hu^+(t)\| | \\ &\leq \|u(t+h) - u(t) - hu^+(t)\| . \end{aligned}$$

Divide by $h > 0$ and take the limit as $h \rightarrow 0^+$. $\square$

Next we compute the subgradient of a convex function of the norm. See (8.6) and (8.7) for examples.

PROPOSITION 8.6. *Let $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be convex, $\Phi(0) = 0$, and V a normed space. Then $f \in V'$ belongs to the subgradient of $\Phi(\|\cdot\|)$ at $u \in V$, i.e.,*

$$(a) \quad f(v-u) \leq \Phi(\|v\|) - \Phi(\|u\|) , \quad v \in V$$

if and only if

$$(b) \quad f(u) = \|f\| \|u\| \quad \text{and} \quad \|f\| \in \partial\Phi(\|u\|) .$$

PROOF. Suppose (a) and set $v = \|u\|w$ for any unit vector w . Thus we obtain $\|u\|f(w) \leq f(u)$, $\|w\| = 1$, and thereby the first part of (b). If $u \neq 0$, set $v = (t/\|u\|)u$ in (a) to obtain

$$\|f\|(t - \|u\|) \leq f(v-u) \leq \Phi(t) - \Phi(\|u\|) , \quad t \in \mathbb{R} .$$

If $u = 0$ this follows by letting $v = tw$ in (a) with $\|w\| = 1$, so (b) follows from (a). Conversely, (b) implies

$$f(v-u) \leq \|f\| (\|v\| - \|u\|) \leq \Phi(\|v\|) - \Phi(\|u\|) , \quad v \in V . \quad \square$$

We note that if $\Phi(r) = r$ we obtain part (d) of Lemma 8.2. Also, if $\Phi(r) = \frac{1}{2}r^2$ and V is a Hilbert space with scalar-product $(\cdot, \cdot)$, then f is given by $f(v) = (u, v)$, $v \in V$, so the derivative of $\frac{1}{2}(\|\cdot\|^2)$ is just the *Riesz map* from V to V' .

DEFINITION. For any normed space V the map $J : V \rightarrow V'$,

$$J(u) = \{ f \in V' : f(u) = \|f\|^2 = \|u\|^2 \}$$

is the *normalized duality map*.

From Proposition 8.6 it follows that $J(u)$ is closed, convex and non-empty. When V' is strictly convex, J is a function (single-valued). More generally, we have the following.

DEFINITION. Let $\xi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be continuous, monotone, surjective and $\xi(0) = 0$. The multi-valued operator $J_\xi : V \rightarrow V'$

$$J_\xi(u) = \{ f \in V' : f(u) = \|f\| \|u\| , \|f\| = \xi(\|u\|) \}$$

is the corresponding *duality map* with *gauge* ξ .

As before, $J_\xi(u)$ is closed, convex, non-empty and J_ξ is a function when V' is strictly convex. To see this, note that for $f, g \in J_\xi(u)$ and $0 < t < 1$ we have $tf + (1-t)g \in V'$ and

$$\langle tf + (1-t)g, u \rangle = (t + (1-t))\xi(\|u\|)\|u\| ,$$

so $J_\xi(u)$ is convex. But by Proposition 8.3.c, it follows that $J_\xi(u)$ is a singleton. Since J_ξ is a G -differential, it is monotone. More precisely, for $u, v \in V$

$$\langle J(u) - J(v), u - v \rangle \geq (\xi(\|u\|) - \xi(\|v\|))(\|u\| - \|v\|),$$

and a corresponding estimate holds in the multi-valued case where V' is not strictly convex.

PROPOSITION 8.7. *If V is a reflexive Banach space and V' is strictly convex, then $J_\xi : V \rightarrow V'$ is monotone and demicontinuous.*

PROOF. Let $u_n \rightarrow u$ in V . Then $\|J_\xi(u_n)\| = \xi(\|u_n\|) \rightarrow \xi(\|u\|)$ since ξ is continuous, and we may suppose $J_\xi(u_n) \rightarrow f$ in V' . Then $\|f\| \leq \xi(\|u\|)$ and

$$f(u) = \lim J(u_n)(u_n) = \lim \xi(\|u_n\|)u_n = \xi(\|u\|)u.$$

Finally, $f(u) \leq \|f\| \|u\|$, so $\|f\| \geq \xi(\|u\|)$ and we have $f = J_\xi(u)$. $\square$

DEFINITION. A normed space V is *uniformly convex* if for each ε , $0 < \varepsilon < 2$, there exists a $\delta > 0$ such that if $\|u\| \leq 1$, $\|v\| \leq 1$ and $\|u - v\| \geq \varepsilon$, then $\|u + v\| \leq 2(1 - \delta)$.

This is equivalent to requiring that if $\|u_n\| \leq 1$, $\|v_n\| \leq 1$ and $\|u_n - v_n\| \rightarrow 2$, then $\|u_n + v_n\| \rightarrow 0$. From the parallelogram law it follows that every Hilbert space is uniformly convex.

PROPOSITION 8.8. *If V' is uniformly convex, then J_ξ is uniformly continuous on each bounded set in V . That is, for $\varepsilon > 0$ and $M > 0$ there is a $\delta > 0$ such that if $\|u\|, \|v\| \leq M$ and $\|u - v\| < \delta$, then $\|J_\xi(u) - J_\xi(v)\| < \varepsilon$.*

PROOF. We give the argument for the normalized case, $\xi(s) = s$, but the general case is similarly obtained. Assume there are sequences in V with $\|u_n\| \leq M$, $\|v_n\| \leq M$, and $\|u_n - v_n\| \rightarrow 0$. If $u_n \rightarrow 0$, then $v_n \rightarrow 0$ and we have $\|Ju_n\| = \|u_n\| \rightarrow 0$ and $\|Jv_n\| = \|v_n\| \rightarrow 0$, so $\|Ju_n - Jv_n\| \rightarrow 0$ and we are done.

Hence, we may assume (by passing to subsequences denoted similarly) that $\|u_n\| \geq \alpha > 0$ and $\|v_n\| \geq \alpha$. Set $x_n = u_n/\|u_n\|$ and $y_n = v_n/\|v_n\|$ so that $\|x_n\| = \|y_n\| = 1$ and

$$x_n - y_n = (u_n - v_n)/\|u_n\| + (\|u_n\|^{-1} - \|v_n\|^{-1})v_n \rightarrow 0.$$

Since $\|J(x_n)\| = \|J(y_n)\| = 1$ we have

$$\begin{aligned} 2 &\geq \|J(x_n) + J(y_n)\| \geq \langle Jx_n + Jy_n, x_n \rangle \\ &= \langle Jx_n, x_n \rangle + \langle Jy_n, y_n \rangle + \langle Jy_n, x_n - y_n \rangle \geq 1 + 1 - \|x_n - y_n\|, \end{aligned}$$

and this shows $\lim_{n \rightarrow \infty} \|J(x_n) + J(y_n)\| = 2$. Since V' is uniformly convex it follows that $\|J(x_n) - J(y_n)\| \rightarrow 0$. We also have

$$Ju_n - Jv_n = (J(x_n) - J(y_n))\|u_n\| + (\|u_n\| - \|v_n\|)J(y_n),$$

and this converges to zero. $\square$

We conclude with some examples of duality maps that are useful in certain applications.

$L^p(\Omega)$. Let $1 \leq p < \infty$ and Ω measurable in $\mathbb{R}^n$. The duality map with gauge $\xi(r) = r^{p-1}$ is the subgradient of the convex function

$$\frac{1}{p} \|v\|_{L^p}^p = \frac{1}{p} \int_{\Omega} |v(x)|^p dx, \quad v \in L^p(\Omega),$$

and by Proposition 8.1 this is characterized by

$$\begin{aligned} f \in J_{\xi}(u) &\iff \int_{\Omega} f(x)u(x) dx = \|f\|_{L^{p'}} \|u\|_{L^p} \text{ and } \|f\|_{L^{p'}} = \|u\|_{L^p}^{p-1}, \\ &\iff f(x) \in |u(x)|^{p-1} \operatorname{sgn}(u(x)), \text{ a.e. } x \in \Omega \end{aligned}$$

for $f \in L^{p'}(\Omega)$ and $u \in L^p(\Omega)$.

$W_0^{1,p}(\Omega)$. Let $1 < p < \infty$ and G a bounded domain in $\mathbb{R}^n$. Then by Poincaré's Lemma 5.1, $W_0^{1,p}(G)$ is a Banach space with norm

$$\|v\| = \left(\sum_{j=1}^n \|\partial_j v\|_{L^p}^p \right)^{1/p}.$$

The duality map with gauge $\xi(r) = r^{p-1}$ is (a subgradient) characterized by

$$\begin{aligned} f \in J_{\xi}(u) &\iff f(u) = \|f\|_{W^{-1,p'}} \|u\|_{W_0^{1,p}} \text{ and } \|f\|_{W^{-1,p'}} = (\|u\|_{W_0^{1,p}})^{p-1} \\ &\iff f = - \sum_{j=1}^n \partial_j (|\partial_j u|^{p-1} \operatorname{sgn}(\partial_j u)) \text{ in } \mathcal{D}^* \end{aligned}$$

for $f \in W^{-1,p'} = (W_0^{1,p})'$, see Section 4, and $u \in W_0^{1,p}(G)$.

II.9. Elliptic Equations in L^1

Let G be a bounded domain in $\mathbb{R}^n$ with smooth boundary ∂G . Consider the linear elliptic differential operator given by

$$Au \equiv - \sum_{i,j=1}^n \partial_j (a_{ij} \partial_i u) + \sum_{i=1}^n \partial_i (a_i u) + au$$

where the coefficients satisfy

$$a_{ij}, \quad a_i \in C^1(\bar{G}) \quad \text{and} \quad a \in L^\infty(G),$$

$$a(x) \geq 0 \quad \text{and} \quad a(x) + \sum_{i=1}^n \partial_i a_i(x) \geq 0,$$

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq c_0 |\xi|^2, \quad \xi \in \mathbb{R}^n$$

for a.e. $x \in G$ and $c_0 > 0$. The meaning of the *Dirichlet problem* for Au is different in each $L^p(G)$. The appropriate definition of $A_p u = f \in L^p(G)$ for $1 \leq p < \infty$ is

that

$$u \in W_0^{1,p}(G) : \int_G \left(\sum_{i,j=1}^n a_{ij} \partial_i u \partial_j v + \left(\sum_{i=1}^n \partial_i (a_i u) + au \right) v \right) = \int_G f v ,$$

$$v \in W_0^{1,p'}(G) .$$

The natural domain is $D(A_p) = \{u \in W_0^{1,p}(G) : A_p u \in L^p(G)\}$. We cite the following fundamental results for these operators

THEOREM 9.1 (AGMON-DOUGLIS-NIRENBERG). *The operator A_p is closed and densely-defined in $L^p(G)$ with domain $D(A_p) = W_0^{1,p}(G) \cap W^{2,p}(G)$ for $1 < p < \infty$, and $(I + \lambda A_p)^{-1}$ is a contraction for each $\lambda > 0$.*

For $1 \leq i \leq n$ we have

$$\int_G \partial_i (a_i u) u = - \int_G a_i u \partial_i u = (1/2) \int_G (\partial_i a_i) u^2 ,$$

so it follows that

$$\int_G \left(\sum_{i=1}^n \partial_i (a_i u) u + au^2 \right) = \int_G \left((1/2) \sum_{i=1}^n \partial_i a_i + a \right) u^2 \geq 0 .$$

This shows $(A_2 u, v)_{L^2}$ is coercive. Thus, we see that $-A_p$ generates a contraction semigroup on $L^p(G)$, and the coercivity estimate shows A_p is a surjection.

The case $p = 1$ is different. The spaces L^1 and L^∞ are not mutually dual so we cannot employ the adjoint arguments used in the proof of Theorem 9.1; also see Lemma 9.1. Since there is not a duality map $L^\infty \rightarrow L^1$, a-priori estimates are more difficult to obtain. As before, we define $D(A_1) = \{u \in W_0^{1,1}(G) : A_1 u \in L^1(G)\}$ where $A_1 u = f \in L^1(G)$ means

$$u \in W_0^{1,1}(G) : \int_G \left(\sum_{i,j=1}^n a_{ij} \partial_i u \partial_j v - \sum_{i=1}^n a_i u \partial_i v + auv \right) = \int_G f v , \quad v \in W_0^{1,\infty}(G) .$$

We shall prove the following.

PROPOSITION 9.1.

- (a) $D(A_1)$ is dense, and $(I + \lambda A_1)^{-1}$ is a contraction on L^1 for each $\lambda > 0$.
- (b) $D(A_1) \subset W_0^{1,q}$ for $1 \leq q < n/(n-1)$ and there is a $c(q) > 0$:
 $c(q) \|u\|_{W^{1,q}} \leq \|A_1 u\|_{L^1}$ for $u \in D(A_1)$.
- (c) A_1 is the L^1 -closure $\bar{A}_2$ of A_2 .
- (d) $\sup_G (I + \lambda A_1)^{-1} f \leq \max\{0, \sup_G f\}$ for each $\lambda > 0$ and $f \in L^1$, that is
 $\|[(I + \lambda A_1)^{-1} f]\|_{L^\infty(G)} \leq \|f^+\|_{L^\infty(G)}$ where $x^+ = \max\{0, x\}$ denotes the positive part of $x \in \mathbb{R}$.

LEMMA 9.1. $A_1 \supset \bar{A}_2$ and $\bar{A}_2$ satisfies (b).

PROOF. From the Sobolev Theorem 4.3, we obtain $W^{2,1} \subset W^{1,q}$, so $D(A_2) = W_0^{1,2} \cap W^{2,2} \subset W^{1,q}$. Also $W_0^{1,2} \subset W_0^{1,1}$ so $A_2 \subset A_1$. That is, we have $D(A_2) \subset D(A_1)$. Consider the adjoint equation

$$- \sum_{i,j=1}^n \partial_i (a_{ij} \partial_j v) - \sum_{i=1}^n a_i \partial_i v + av = - \sum_{i=1}^n \partial_i h_i .$$

Stampacchia (1963) showed there is a unique solution $v \in H_0^1 \cap L^\infty$ if each $h_i \in L^p$, $p > n$: for each $w \in H_0^1$

$$\int_G \left(\sum_{i,j=1}^n a_{ij} \partial_j v \partial_i w - \sum a_i \partial_i v w + a v w \right) = \int_G \sum_{i=1}^n h_i \partial_i w$$

and it satisfies

$$\|v\|_{L^\infty} \leq C \sum_{i=1}^n \|h_i\|_{L^p}.$$

By choosing $w = u \in D(A_2)$ we obtain

$$\sum_{i=1}^n \int_G h_i \partial_i u = \int_G v A_2 u \leq \|v\|_{L^\infty} \|A_2 u\|_{L^1} \leq C \sum_{i=1}^n \|h_i\|_{L^p} \|A_2 u\|_{L^1}.$$

Since $h_1, \dots, h_n$ is arbitrary in $(L^p)^n$ it follows that

$$\sum_{i=1}^n \|\partial_i u\|_{L^q} \leq C \|A_2 u\|_{L^1}, \quad q = p/(p-1) < n/(n-1).$$

Since the graph of A_1 is closed in $W^{1,1} \times L^1$, it follows that $\bar{A}_2 \subset A_1$. $\square$

LEMMA 9.2. *Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be Lipschitz, monotone and $\varphi(0) = 0$. Then*

$$(A_2 u, \varphi_\varepsilon(u))_{L^2} \geq 0 \quad \text{for } u \in D(A_2).$$

PROOF. We have

$$(A_2 u, \varphi(u))_{L^2} = \int_G \left(\sum_{i,j=1}^n (a_{ij} \partial_i u \partial_j u) \varphi'(u) + \sum_{i=1}^n \partial_i (a_i u) \varphi(u) + a u \varphi(u) \right)$$

and the first term is non-negative by ellipticity of A and monotonicity of φ . To estimate the other terms set $\zeta(t) = \int_0^t s \varphi'(s) ds$ and note $0 \leq \zeta(t) = t \varphi(t) - \int_0^t \varphi \leq t \varphi(t)$. The sum of the second and third terms is

$$\begin{aligned} \int_G \left(- \sum_{i=1}^n a_i u \varphi'(u) \partial_i u + a u \varphi(u) \right) &\geq \int_G \left(- \sum_{i=1}^n a_i \partial_i \zeta(u) + a \zeta(u) \right) \\ &= \int_G \left(\sum_{i=1}^n \partial_i a_i + a \right) \zeta(u) \geq 0. \end{aligned} \quad \square$$

LEMMA 9.3. $\bar{A}_2$ satisfies (a).

PROOF. Let $f \in L^2$, $u = (I + \lambda A_2)^{-1} f$ so $u + \lambda A_2 u = f$. For $\varepsilon > 0$ define $\varphi_\varepsilon(u) = \frac{1}{\varepsilon} (u - (I + \varepsilon \operatorname{sgn})^{-1} u) = \operatorname{sgn}(u)$ for $|u| \geq \varepsilon$ and $= u/\varepsilon$ for $|u| \leq \varepsilon$. Multiply by $\varphi_\varepsilon(u)$ and integrate: $\int_G u \varphi_\varepsilon(u) \leq \int_G f \varphi_\varepsilon(u) \leq \|f\|_{L^1}$. By letting $\varepsilon \rightarrow 0^+$ we obtain $\|u\|_{L^1} \leq \|f\|_{L^1}$, hence, $(I + \lambda A_2)^{-1}$ is a contraction on L^2 .

Finally note that if $f_n \in L^2$ and $f_n \rightarrow f \in L^1$ with convergence in L^1 , then $u_n \equiv (I + \lambda A_2)^{-1} f_n$ converges in L^1 to $u = (I + \lambda \bar{A}_2)^{-1} f$ and $\|u\|_{L^1} \leq \|f\|_{L^1}$. $\square$

LEMMA 9.4. A_1 is one-to-one.

PROOF. For each $g \in L^p$, $p > n$, there is a solution v of the adjoint problem $v \in W_0^{1,p} \cap W^{2,p}$:

$$\int_G \left(\sum_{i,j=1}^n a_{ij} \partial_j v \partial_i w - \sum_{i=1}^n a_i \partial_i v w + a v w \right) = \int_G g w, \quad w \in C_0^\infty.$$

Since $W^{2,p} \subset C^1(\bar{G})$ by Theorem 4.3, we can let $w \in W_0^{1,1}$. Taking $w = u$ where $A_1 u = 0$ we obtain $\int_G g u = \int_G v A_1 u = 0$ for g as above, so $u = 0$. $\square$

COROLLARY 9.1. $I + \lambda A_1$ is one-to-one for each $\lambda > 0$.

We have shown $\bar{A}_2 \subset A_1$, $I + \lambda \bar{A}_2$ is onto and $I + \lambda A_1$ is one-to-one. Thus $\bar{A}_2 = A_1$ and all of Proposition 9.1 is proved except (d). The proof of this maximum principle is the same as the L^1 -estimate in (a).

LEMMA 9.2A. For $\varepsilon > 0$ define $\varphi_\varepsilon^+(u) = \frac{1}{\varepsilon} (u - (I + \varepsilon \operatorname{sgn}^+)^{-1} u) = \operatorname{sgn}^+(u)$ for $u \geq \varepsilon$ or $u \leq 0$ and $\varphi_\varepsilon^+(u) = u/\varepsilon$ for $0 \leq u \leq \varepsilon$. Then

$$(A_2 u, \varphi_\varepsilon^+(u - k))_{L^2} \geq 0 \quad \text{for } u \in D(A_2), \quad k \geq 0.$$

LEMMA 9.3A. A_2 satisfies (d).

PROOF. If $(I + \lambda A_2)u = f \in L^2$ then $u - k + \lambda A_2 u = f - k$. Multiply by $\varphi_\varepsilon^+(u - k)$, integrate and let $\varepsilon \rightarrow 0^+$ to obtain

$$\int_G (u - k)^+ \leq \int_G (f^+ - k)^+,$$

where $r^+ = r \operatorname{sgn}^+(r)$. Setting $k = \sup f^+$ shows $(u - k)^+ = 0$, and so we obtain

$$\|u^+\|_{L^\infty} \leq \|f^+\|_{L^\infty}. \quad \square$$

COROLLARY 9.2. A_1 satisfies (d).

PROOF. Let $f \in L^1$ and choose $f_n \in L^2$ such that $f_n \rightarrow f$ in L^1 , $f_n(x) \rightarrow f(x)$, $f_n(x) \leq f(x)^+$, and $u_n(x) = (I + \lambda A_2)^{-1} f_n(x) \rightarrow u(x)$ a.e. $x \in G$. Then let $n \rightarrow \infty$ in $u_n(x) \leq \|f_n^+\|_{L^\infty} \leq \|f^+\|_{L^\infty}$. $\square$

We shall extend Lemma 9.2 and Lemma 9.2A to more general monotone functions. Also, the *negative part* of any $x \in \mathbb{R}$ will be denoted by $x^- = \min\{0, x\}$. A useful estimate is given by the following.

PROPOSITION 9.2. Let $T : L^1(G) \rightarrow L^1(G)$ satisfy

- (a) $\|Tu - Tv\|_{L^1} \leq \|u - v\|_{L^1}$, $u, v \in L^1$, and
- (b) $-\|u^-\|_{L^\infty} \leq Tu(x) \leq \|u^+\|_{L^\infty}$ a.e. $x \in G$. Let $j : \mathbb{R} \rightarrow \mathbb{R}_+$ be convex, lower-semi-continuous and $j(0) = 0$. Then for each $u \in L^1(G)$

$$\int_G j(Tu(x)) dx \leq \int_G j(u(x)) dx.$$

PROOF. Consider the special case $j_1(r) = (r - t)^+$ with $t \geq 0$. Set $v(x) = \min\{u(x), t\}$ so $v \in L^1$ and $|u(x) - v(x)| = (u(x) - t)^+$. Also $Tv(x) \leq \|v^+\|_{L^\infty} \leq t$ so

$$(Tu(x) - t)^+ \leq (Tu(x) - Tv(x))^+ \leq |Tu(x) - Tv(x)| ,$$

hence

$$\int_G (Tu(x) - t)^+ dx \leq \|u - v\|_{L^1} = \int_G (u(x) - t)^+ dx$$

as desired. The same holds for $u \mapsto -T(-u)$ and so for $t \leq 0$ we have

$$\int_G (-Tu(x) + t)^+ dx \leq \int_G (-u(x) + t)^+ dx .$$

These may be summarized by

$$(9.1) \quad \int_G [t(Tu(x) - t)]^+ dx \leq \int_G [t(u(x) - t)]^+ dx , \quad t \in \mathbb{R} .$$

Consider the smooth case with $0 \leq j'' \in L^\infty(\mathbb{R})$ and, hence, $j'(0) = 0$. First check that

$$j(s) = \int_{-\infty}^{\infty} j''(t) [\operatorname{sgn}(t)(s - t)]^+ dt .$$

Then it follows that we can multiply (9.1) by $j''(t)/|t|$ and integrate via Tonelli to obtain the desired estimate.

Finally, for any function j as given in the Proposition 9.2 we construct a sequence of smooth functions as above, e.g., $j_\varepsilon(r) = \inf_{t \in \mathbb{R}} \{(1/2\varepsilon)|r - t|^2 + j(t)\}$, which converge monotonically to j from below. Then

$$\int_G j_\varepsilon(Tu) \leq \int_G j_\varepsilon(u) \leq \int_G j(u) , \quad \varepsilon > 0 ,$$

and this implies the desired estimate. $\square$

COROLLARY 9.3. $\|Tu\|_{L^p} \leq \|u\|_{L^p}$, $\|Tu^+\|_{L^p} \leq \|u^+\|_{L^p}$ for $p \geq 1$.

PROOF. Take $j(r) = r^p$ and $(r^+)^p$. $\square$

The next result is central and roughly asserts that $\int_G Au \cdot \beta(u) dx \geq 0$ for any maximal monotone graph $\beta \subset \mathbb{R} \times \mathbb{R}$ which contains the origin.

PROPOSITION 9.3. *Let β be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ and $0 \in \beta(0)$; let A satisfy (a) and (d) of Proposition 9.1; let $1 \leq p \leq \infty$ and $1/p + 1/p' = 1$. Then for each pair $u \in L^p, v \in L^{p'}$ with $Au \in L^p$ and $v(x) \in \beta(u(x))$ a.e. $x \in G$*

$$\int_G Au(x)v(x) dx \geq 0 .$$

PROOF. Let the convex lower-semi-continuous $j : \mathbb{R} \rightarrow \mathbb{R}_\infty$ be the indefinite integral of β with $j(0) = 0$; that is, $\partial j = \beta$. (See Example 8.A.) For each $\lambda > 0$ set $T_\lambda = (I + \lambda A)^{-1}$, so (a) and (b) of Proposition 9.2 are satisfied. Note that $\lambda T_\lambda A = I - T_\lambda$ on $D(A)$ so $T_\lambda \rightarrow I$ pointwise on L^1 . Since $v(x) \in \partial j(u(x))$, a.e. $x \in G$, we have $v(x)(0 - u(x)) \leq j(0) - j(u(x))$, hence, $0 \leq j(u(x)) \leq v(x)u(x)$, so $j \circ u \in L^1$. Likewise

$$-\lambda v(x)(T_\lambda Au(x)) = v(x)(T_\lambda u(x) - u(x)) \leq j(T_\lambda u(x)) - j(u(x)) , \quad \text{a.e. } x \in G ,$$

so from Proposition 9.2 we obtain

$$\int_G T_\lambda Au(x)v(x) dx \geq 0, \quad \lambda > 0.$$

If $p = 1$ the result follows from the strong convergence $T_\lambda Au \rightarrow Au$ in L^1 . If $1 < p < \infty$ then Proposition 9.2 shows $\|T_\lambda Au\|_{L^p} \leq \|Au\|_{L^p}$ and the result follows from the weak convergence $T_\lambda Au \rightharpoonup Au$ in L^p . If $p = \infty$ then $\|T_\lambda Au\|_{L^\infty} \leq \|Au\|_{L^\infty}$; a subsequence a.e. convergent gives the result by dominated convergence. $\square$

We have shown that the operator A_1 satisfies the hypotheses on the abstract linear operator A in $L^1(G)$ in the following result. As a consequence, for any maximal monotone graph β with $0 \in \beta(0)$ it follows by setting $\beta = \alpha^{-1}$ that the composite nonlinear operator $A_1 \circ \beta$ is m -accretive in $L^1(G)$, and its resolvent equation satisfies some useful *comparison estimates*. Additional examples will be given below.

THEOREM 9.2 (BREZIS-STRAUSS). *Let α be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ and $0 \in \alpha(0)$. Let $A : D(A) \rightarrow L^1(G)$ be linear and satisfy the following:*

- (i) *$D(A)$ is dense and $(I + \lambda A)^{-1}$ is a contraction in L^1 for each $\lambda > 0$;*
- (ii) *$\sup_G (I + \lambda A)^{-1} f \leq (\sup_G f)^+ = \|f^+\|_{L^\infty}$ for $f \in L^1$ and $\lambda > 0$;*
- (iii) *there is a $c > 0$ such that*

$$c\|u\|_{L^1} \leq \|Au\|_{L^1} \quad \text{for } u \in D(A).$$

Then for each $f \in L^1$ there is a unique pair $u \in D(A)$, $v \in L^1$ such that

$$(9.2) \quad Au + v = f \quad \text{and} \quad v(x) \in \alpha(u(x)), \quad \text{a.e. } x \in G.$$

If u_1, v_1 and u_2, v_2 are solutions corresponding to f_1, f_2 as above, then

$$(9.3) \quad \|(v_1 - v_2)^+\|_{L^1} \leq \|(f_1 - f_2)^+\|_{L^1}, \quad \|(v_1 - v_2)^-\|_{L^1} \leq \|(f_1 - f_2)^-\|_{L^1},$$

and, hence,

$$(9.4) \quad \|v_1 - v_2\|_{L^1} \leq \|f_1 - f_2\|_{L^1}.$$

If $f_1 \geq f_2$ a.e. then $v_1 \geq v_2$ a.e. on G .

PROOF. The last claim follows from the second estimate in (9.3), and (9.4) follows by adding (9.3). By symmetry it suffices to verify the first estimate in (9.3). Multiply the equation

$$A(u_1 - u_2) + (v_1 - v_2) = f_1 - f_2$$

by $v(x) \equiv \text{sgn}_0^+(u_1(x) - u_2(x) + v_1(x) - v_2(x))$ where $\text{sgn}_0^+(r) = 1$ for $r > 0$ and $= 0$ otherwise. Note $v(x) \in \text{sgn}^+(u_1(x) - u_2(x)) \cap \text{sgn}^+(v_1(x) - v_2(x))$ since α is monotone. From Proposition 9.3 we obtain

$$\|(v_1 - v_2)^+\|_{L^1} = \int_G (v_1 - v_2)v \leq \int_G (f_1 - f_2)v \leq \int_G (f_1 - f_2)^+ v \leq \|(f_1 - f_2)^+\|_{L^1}.$$

It remains to show $Rg(A + \alpha)$ is all of L^1 . To show the range is closed, let $Au_n + v_n = f_n \rightarrow f$ in L^1 with $v_n \in \alpha(u_n)$ for $n \geq 1$. Then (9.4) implies $v_n \rightarrow v$ in L^1 and (iii) implies $u_n \rightarrow u$ in L^1 with $Au + v = f$. Since $(I + \alpha)^{-1}$ is a Lipschitz function we may take the limit in $u_n = (I + \alpha)^{-1}(u_n + v_n)$ to obtain

$u = (I + \alpha)^{-1}(u + v)$, hence, $v \in \alpha(u)$. It suffices now to show $Rg(A + \alpha)$ is dense in L^1 .

In order to solve (9.2) we shall *regularize* both operators, replacing A by $A + \varepsilon I$ and α^{-1} by $\alpha^{-1} + \lambda I$ where $\varepsilon, \lambda > 0$. Note the equivalence of $(\lambda I + \alpha^{-1})(y) \ni x$, $y \in \alpha(x - \lambda y)$, $x - \lambda y + \lambda \alpha(x - \lambda y) \ni x$ and $y = (1/\lambda)(I - (I + \lambda \alpha)^{-1})(x)$. Thus we set $\alpha_\lambda \equiv (1/\lambda)(I - (I + \lambda \alpha)^{-1})$, the monotone Lipschitz (Yosida) approximation of α , and consider the equation

$$(9.5) \quad \varepsilon u + Au + \alpha_\lambda(u) = f .$$

This is equivalent to

$$u = (1 + \lambda \varepsilon)^{-1} (I + (\lambda/1 + \varepsilon \lambda)A)^{-1} (\lambda f + (I + \lambda \alpha)^{-1}u) ,$$

and so this strict contraction on the right side guarantees there is a solution in L^1 . But we shall need more ... estimates in L^2 ... so we consider $L^1 \cap L^\infty$ with the norm $\|u\| = \|u\|_{L^1} + \|u\|_{L^\infty}$. If $f \in L^1 \cap L^\infty$ and $\varepsilon > 0$ is fixed, then for each $\lambda > 0$ there is a solution $u_\lambda \in L^1 \cap L^\infty$ of (9.5) obtained as a fixed-point in this space, and it satisfies $\|u_\lambda\| \leq (1 + \lambda \varepsilon)^{-1} (\lambda \|f\| + \|u_\lambda\|)$, hence, $\|u_\lambda\| \leq (1/\varepsilon) \|f\|$. Next multiply (9.5) by $\text{sgn}(u_\lambda)$ and integrate to obtain from Proposition 9.3 $\|\alpha_\lambda(u_\lambda)\|_{L^1} \leq \|f\|_{L^1}$. Similarly get an L^p estimate and let $p \rightarrow \infty$ to give $\|\alpha_\lambda(u_\lambda)\|_{L^\infty} \leq \|f\|_{L^\infty}$ for each $\lambda > 0$. This shows that $\{\alpha_\lambda(u_\lambda)\}$ is bounded in L^2 , and we next deduce that $\{u_\lambda\}$ is Cauchy in L^2 . Take the difference of (9.5) at λ, μ and scalar product with $u_\lambda - u_\mu$ to obtain from Proposition 9.3

$$\varepsilon \|u_\lambda - u_\mu\|_{L^2}^2 + (\alpha_\lambda(u_\lambda) - \alpha_\mu(u_\mu), u_\lambda - u_\mu)_{L^2} \leq 0 .$$

Substitute in the right term

$$u_\lambda - u_\mu = (I - (I + \lambda \alpha)^{-1})u_\lambda + ((I + \lambda \alpha)^{-1}u_\lambda - (I + \mu \alpha)^{-1}u_\mu) - (I - (I + \mu \alpha)^{-1})u_\mu$$

and note the middle product is non-negative because $\alpha_\lambda \in \alpha(I + \lambda \alpha)^{-1}$; this gives

$$\varepsilon \|u_\lambda - u_\mu\|_{L^2}^2 + (\alpha_\lambda(u_\lambda) - \alpha_\mu(u_\mu), \lambda \alpha_\lambda(u_\lambda) - \mu \alpha_\mu(u_\mu)) \leq 0 .$$

Since $\|\alpha_\lambda(u_\lambda)\|_{L^2}$ is bounded, it follows that $\{u_\lambda\}$ is Cauchy in L^2 . From Lemma 9.5 below it also follows that $\{\alpha_\lambda(u_\lambda)\}$ is Cauchy in L^2 . By passing to a subsequence we obtain $u_\lambda \rightarrow u$ and $\alpha_\lambda(u_\lambda) \rightarrow v$ in $L^2(G)$ with $u, v \in L^1 \cap L^\infty$. As before we find $v(x) \in \alpha(u(x))$ a.e. and $(\varepsilon I + A)u_\lambda \rightarrow f - v$ in $L^2(G)$. Set $w = (\varepsilon I + A)^{-1}(f - v)$; then $w \in D(A) \cap L^\infty$ and $(\varepsilon I + A)(u_\lambda - w) \rightarrow 0$ and $u_\lambda \rightarrow w = u$. Thus

$$\varepsilon u + Au + v = f , \quad v \in \alpha(u) .$$

Finally, let $f \in L^1$ and $f_\varepsilon \in L^1 \cap L^\infty$ with $f_\varepsilon \rightarrow f$ in L^1 . For $\varepsilon > 0$ let u_ε be the solution as above of

$$\varepsilon u_\varepsilon + Au_\varepsilon + v_\varepsilon = f_\varepsilon , \quad v_\varepsilon \in \alpha(u_\varepsilon) .$$

As before we get $\varepsilon \|u_\varepsilon\|_{L^1} + \|v_\varepsilon\|_{L^1} \leq \|f_\varepsilon\|_{L^1}$ from Proposition 9.3 and then (iii) gives

$$c \|u_\varepsilon\|_{L^1} \leq 2 \|f_\varepsilon\|_{L^1} \leq 2 \|f\|_{L^1} ,$$

hence, $\varepsilon u_\varepsilon \rightarrow 0$ and $f = \lim_{\varepsilon \rightarrow 0} (f_\varepsilon - \varepsilon u_\varepsilon)$ belongs to the closure of $Rg(A + \alpha)$. This finishes the proof of Theorem 9.2 except for the following general result. $\square$

LEMMA 9.5. *Let $\{z_\lambda\}$ be given in a scalar-product space for $\lambda > 0$ and assume*

$$(z_\lambda - z_\mu, \lambda z_\lambda - \mu z_\mu) \leq 0, \quad \lambda, \mu > 0.$$

Then $\lambda \mapsto \|z_\lambda\|$ is monotone decreasing; if $\{z_\lambda\}$ is bounded then it is Cauchy for $\lambda \rightarrow 0^+$.

PROOF. From the calculation

$$0 \geq 2(z_\lambda - z_\mu, \lambda z_\lambda - \mu z_\mu) = (\lambda + \mu)\|z_\lambda - z_\mu\|^2 + (\lambda - \mu)(\|z_\lambda\|^2 - \|z_\mu\|^2)$$

we obtain

$$(\lambda + \mu)\|z_\lambda - z_\mu\|^2 \leq (\lambda - \mu)(\|z_\mu\|^2 - \|z_\lambda\|^2)$$

so $\lambda > \mu$ implies $\|z_\lambda\| \leq \|z_\mu\|$. Also we have

$$\|z_\lambda - z_\mu\|^2 \leq \|z_\mu\|^2 - \|z_\lambda\|^2, \quad \lambda > \mu > 0,$$

so boundedness implies the Cauchy condition. $\square$

EXAMPLE 9.A. This first order operator is the L^1 realization of Example I.4.B. Set $D(A) = \{v \in W^{1,1}(a, b) : v(a) = cv(b)\}$, where $a < b$ and $0 \leq c < 1$. Define $A = \partial$ on $D(A)$. It is easy to check that there is a solution $u \in D(A)$ of $(I + \lambda A)(u) = f$ for each $f \in L^1(a, b)$ and $\lambda > 0$. For any such solution, we multiply the equation by $\text{sgn}(u)$ and integrate to get $\|u\|_{L^1(a, b)} \leq \|f\|_{L^1(a, b)}$, so (i) holds. Multiply the identity

$$u(x) - k + \lambda Au(x) = f(x) - k$$

by $\text{sgn}^+(u(x) - k)$, integrate, and set $k = \|f^+\|_{L^\infty(a, b)}$ to obtain

$$\|(u - k)^+\|_{L^1(a, b)} + (u(b) - k)^+ - (cu(b) - k)^+ \leq 0.$$

Since $k \geq 0$, $c \geq 0$ and the *positive part* function $(\cdot)^+$ is monotone, we have

$$\begin{aligned} (u(b) - k)^+ - (cu(b) - k)^+ &\geq (u(b) - k)^+ - (cu(b) - ck)^+ \\ &\geq (1 - c)(u(b) - k)^+ \geq 0, \end{aligned}$$

so $(u - k)^+ = 0$ and we have (ii). To check (iii), we let $Au = f$. Multiply by $\text{sgn}(u)$ and integrate to get

$$|u(x)| \leq |u(a)| + \|f\|_{L^1(a, b)}, \quad a \leq x \leq b.$$

Integrating $Au = f$ and using the boundary condition give $u(a) = \frac{c}{1-c}\|f\|_{L^1(a, b)}$, so with the above we obtain (iii) from

$$|u(x)| \leq \frac{1}{1-c}\|f\|_{L^1(a, b)}, \quad a \leq x \leq b.$$

Theorem 9.2 asserts that for any maximal monotone β in $\mathbb{R} \times \mathbb{R}$ with $0 \leq \beta(0)$, the operator $\partial \circ \beta$ is m -accretive in $L^1(a, b)$. Specifically, if $f \in L^1(a, b)$ there is a unique pair

$$v \in L^1(a, b), \quad u \in W^{1,1}(a, b),$$

for which $u(a) = cu(b)$ and

$$v(x) + \partial u(x) = f(x), \quad u(x) \in \beta(v(x)), \quad \text{a.e. } x \in (a, b),$$

and the mapping $f \mapsto v$ is an $L^1(a, b)$ -contraction. This operator will arise in the study of *scalar conservation laws*. Other such first order operators can be so constructed; one can delete the second order terms from A_1 above, for example.

EXAMPLE 9.B. We record here a special case of Proposition 9.1. Let β be given as above. Then for each $f \in L^1(G)$ there is a unique pair

$$v \in L^1(G) \quad , \quad u \in W_0^{1,1}(G)$$

for which the Laplacian $\Delta u \in L^1(G)$ and

$$v(x) - \Delta u(x) = f(x) \quad , \quad u(x) \in \beta(v(x)) \quad , \quad \text{a.e. } x \in G \quad ,$$

and the mapping $f \mapsto v$ is a contraction in $L^1(G)$. The operator $-\Delta \circ \beta$ corresponds to the stationary problem for the *porous medium equation*, and one can construct similar operators by varying either the linear elliptic part, $-\Delta$, as above or the boundary conditions.

CHAPTER III

Nonlinear Evolution Problems

III.1. Vector-valued Functions

We shall develop here some of the calculus notions that we shall use hereafter. These consist of an extension to vector-valued functions of the integral and the derivative. Here the functions take values in a Banach space. The domain of these functions can be a general measure space, without adding any technical difficulties to the development of the Lebesgue integration theory, so we consider this general situation initially. For the differentiation theory we restrict our attention to those functions whose domain is an interval of real numbers, so the classical formulation of the derivative as a limit of difference quotients is valid and intuitive.

Let (S, M, μ) be a measure space and B a Banach space with norm $\|\cdot\|$ and dual B' . A step function $f : S \rightarrow B$ (i.e., a finite-valued function) is *measurable* if $f^{-1}(x) \in M$ for each $x \in B$ and it is *integrable* if also each $\mu(f^{-1}(x)) < \infty$ for $x \neq 0$. Then we define the integral $\int_S f \, d\mu = \sum_{x \in B} \mu(f^{-1}(x))x$ as the indicated finite linear combination of vectors in the range, $Rg(f)$.

A function $f : S \rightarrow B$ is *measurable* if there is a sequence f_n of measurable step functions for which $f_n(s) \rightarrow f(s)$ in B for a.e. $s \in S$.

THEOREM 1.1 (PETTIS). *The function $f : S \rightarrow B$ is measurable if and only if (i) f is a.e. separably valued (some subset of full measure in S is carried by f to a separable subset of B) and (ii) f is weakly-measurable (for each $g \in B'$ the real-valued $s \mapsto g(f(s)) : S \rightarrow \mathbb{R}$ is measurable).*

A function $f : S \rightarrow B$ is *integrable* if there is a sequence of integrable step functions f_n such that $\lim_{n \rightarrow \infty} \int_S \|f_n(s) - f(s)\| \, d\mu = 0$ and each integrand is integrable. Then $\int_S f_n \, d\mu$ converges in B to a limit which is the same for any such sequence; this limit is denoted by $\int_S f \, d\mu$.

THEOREM 1.2 (BOCHNER). *The function $f : S \rightarrow B$ is integrable if and only if f is measurable and $s \mapsto \|f(s)\|$ is integrable.*

THEOREM 1.3 (FATOU). *Let f_n be integrable and $f_n(s) \rightarrow f(s)$ (weakly) at a.e. $s \in S$ with $\{\int_S \|f_n\| \, d\mu\}$ bounded. Then f is integrable and $\int_S \|f\| \, d\mu \leq \liminf \int_S \|f_n\| \, d\mu$.*

THEOREM 1.4 (LEBESGUE). *Let f_n be integrable and $f_n(s) \rightarrow f(s)$ (strongly) at a.e. $s \in S$ and suppose there is an integrable $g : S \rightarrow \mathbb{R}$ such that $\|f_n(s)\| \leq g(s)$ for all n and a.e. $s \in S$. Then f is integrable and $\int_S \|f_n - f\| \, d\mu \rightarrow 0$, hence, $\int_S f_n \, d\mu \rightarrow \int_S f \, d\mu$ in B .*

Let $1 \leq p \leq +\infty$; we denote by $L^p(S, B)$ the space of (equivalence classes of) measurable functions $f : S \rightarrow B$ such that $\|f(\cdot)\|$ belongs to $L^p(S, \mathbb{R})$. With the

respective norms

$$\|f\|_{L^p} \equiv \left(\int_S \|f(s)\|^p d\mu \right)^{1/p}, \quad 1 \leq p < \infty,$$

$$\|f\|_{L^\infty} \equiv \text{ess sup}\{\|f(s)\| : s \in S\},$$

each $L^p(S, B)$ is a Banach space. If B is separable and $1 \leq p < \infty$, then $L^p(S, B)$ is separable.

THEOREM 1.5 (PHILLIPS). *Let μ be σ -finite, $1 < p < \infty$ and $1/p' + 1/p = 1$, and B reflexive. Then the dual $L^p(S, B)'$ can be identified with $L^{p'}(S, B')$.*

To be precise, for each $g \in L^p(S, B)'$ there is a $f \in L^{p'}(S, B')$ such that

$$g(x) = \int_S \langle f(s), x(s) \rangle d\mu(s), \quad x \in L^p(S, B).$$

Specifically, the real-valued $s \mapsto \langle f(s), x(s) \rangle$ is integrable. Finally, if B is Hilbert space, so also is $L^2(S, B)$ with the scalar product

$$(f, g)_{L^2(S, B)} = \int_S (f(s), g(s))_B d\mu, \quad f, g \in L^2(S, B).$$

Consider the case $S = (0, T)$, an interval of length $T > 0$. Let $f \in L^1(0, T; B)$ and define the primitive

$$F(t) = \int_0^t f(s) ds, \quad 0 \leq t \leq T.$$

THEOREM 1.6 (LEBESGUE). *At a.e., $t \in (0, T)$, F is (strongly) differentiable with $F'(t) \equiv \lim_{h \rightarrow 0} h^{-1}(F(t+h) - F(t)) = f(t)$.*

See [1], [7], [27], [59] for proofs and additional information on the above.

Define $W^{1,p}(0, T; B)$ to be the set of functions $f : [0, T] \rightarrow B$ such that for some $g \in L^p(0, T; B)$

$$f(t) = f(0) + \int_0^t g(s) ds, \quad t \in [0, T].$$

More generally, a function $f : [0, T] \rightarrow B$ is *absolutely continuous* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for each sequence of disjoint intervals (a_n, b_n) in $[0, T]$ verifying $\sum_n (b_n - a_n) < \delta$ there follows $\sum_n \|f(b_n) - f(a_n)\| < \varepsilon$. If $f \in W^{1,1}(0, T; B)$, then $\|f(t) - f(s)\| \leq \int_s^t \|f'(t)\| dt$, so f is absolutely continuous. Likewise, if $f \in W^{1,p}(0, T; B)$ with $1 < p \leq \infty$, then $\|f(t) - f(s)\| \leq C|t - s|^{1/p'}$ for $t, s \in [0, T]$.

PROPOSITION 1.1. *The function $f : [0, T] \rightarrow B$ is weakly absolutely continuous (i.e., for each $\varphi \in B'$ the real-valued $t \mapsto \varphi(f(t))$ is AC), f is weakly differentiable, and $f' \in L^p(0, T; B)$ if and only if $f \in W^{1,p}(0, T; B)$.*

PROOF. The “if” part is clear from above. Conversely, set $F(t) \equiv f(0) + \int_0^t f'(s) ds$, $t \in [0, T]$. For each $\varphi \in B'$, $\varphi(f(\cdot))$ is differentiable a.e. and

$$\frac{d}{dt} \varphi(f(t)) = \varphi(f'(t)) \quad , \quad \text{a.e. } t \in [0, T] \quad .$$

By weak absolute continuity there follows

$$\varphi(f(t)) = \varphi(f(0)) + \int_0^t \varphi(f'(s)) ds = \varphi(F(t)) \quad ,$$

for all $\varphi \in B'$ and all $t \in [0, T]$. Thus $F = f \in W^{1,p}$. $\square$

The preceding proof was made easy by the requirement that the weak derivative belong to L^p . A much deeper result is the following.

THEOREM 1.7 (KŌMURA). *Assume B is reflexive and $f : [0, T] \rightarrow B$ is absolutely continuous. Then f is (strongly) differentiable a.e., $f' \in L^1(0, T; B)$ and $f(t) = f(0) + \int_0^t f' \quad , \quad 0 \leq t \leq T$.*

Let V be a Banach space. We shall consider solutions u of evolution equations where $u \in L^p(0, T; V)$ and the equation holds in the space $L^{p'}(0, T; V')$. Then the strong derivative satisfies $u' \in L^{p'}(0, T; V')$, so it follows that u is absolutely continuous with values in V' ; thus it is meaningful to say initial conditions are attained in the sense of V' . A much better description of initial or pointwise values is obtained from the following.

Let H be a Hilbert space which is identified with its dual $H \cong H'$ by the Riesz map and in which V is dense and continuously embedded. Then we have $V \hookrightarrow H \hookrightarrow V'$ with the equation

$$f(v) = (f, v)_H \quad \text{for } f \in H \subset V' \quad , \quad v \in V \quad .$$

Consider the Banach space

$$W_p(0, T) \equiv \{u \in L^p(0, T; V) : u' \in L^{p'}(0, T; V')\}$$

with the norm

$$\|u\|_{W_p} = \|u\|_{L_p} + \|u'\|_{L_{p'}} \quad .$$

Let $a < 0 < T < b$; we shall construct an extension of each $u \in W_p(0, T)$ to $\tilde{u} \in W_p(a, b)$. First, extend u to $(a, 0)$ and (T, b) (e.g., by symmetry). Let $\theta \in C_0^\infty(a, b)$ with $\theta = 1$ on $(0, T)$; define $\tilde{u} = u \cdot \theta$ and note that $\tilde{u} \in W_p(a, b)$ and $\tilde{u} = u$ on $(0, T)$. Also, $\|u\|_{W_p(0, T)} \leq \|\tilde{u}\|_{W_p(a, b)} \leq C(\theta)\|u\|_{W_p(0, T)}$ where $C(\theta)$ depends on θ , and $\tilde{u}$ equals 0 in a neighborhood of a . Next, we regularize $\tilde{u}$ by the mollifier $u_m(t) = \int \tilde{u}(s)\rho_m(t-s)ds$, $m \geq 1$, where $\rho_m(s) = m\rho(ms)$ and $\rho \in C_0^\infty(-1, 1)$, $\rho \geq 0$, $\int \rho = 1$. As in Proposition II.3.2 it follows that $u_m \in C_0^\infty((a, b), V)$ for sufficiently large m and $u_m \rightarrow \tilde{u}$ in $W_p(a, b)$ with $\|u_m\|_{W_p} \leq \|\tilde{u}\|_{W_p}$. By the preceding identification of spaces we have $(1/2) \frac{d}{dt} |u_m(t)|_H^2 = (u'_m(t), u_m(t))_H =$

$u'_m(t)(u_m(t))$, and this yields

$$\begin{aligned} (1/2)|u_m(t)|_H^2 &= \int_a^t u'_m(s)(u_m(s)) \, ds \\ &\leq \int_a^t \|u'_m(s)\|_{V'} \|u_m(s)\|_V \, ds \\ &\leq \|u_m\|_{W_p(a,b)}^2, \quad a \leq t \leq b. \end{aligned}$$

Since $\{u_m\}$ is Cauchy in $W_p(a, b)$, such an estimate on differences $u_m - u_n$ from the sequence shows that it converges (uniformly) to $\tilde{u}$ in $C([a, b], H)$. Thus we obtain the following.

PROPOSITION 1.2. *Let the Banach space V be dense and continuously embedded in the Hilbert space H ; identify $H = H'$ so that $V \hookrightarrow H \hookrightarrow V'$. The Banach space $W_p(0, T) \equiv \{u \in L^p(0, T; V) : u' \in L^{p'}(0, T; V')\}$ is contained in $C([0, T], H)$. Moreover, if $u \in W_p(0, T)$ then $|u(\cdot)|_H^2$ is absolutely continuous on $[0, T]$,*

$$\frac{d}{dt}|u(t)|_H^2 = 2u'(t)(u(t)) \quad \text{a.e.} \quad t \in [0, T],$$

and there is a constant C for which

$$\|u\|_{C([0, T], H)} \leq C\|u\|_{W_p(0, T)}, \quad u \in W_p.$$

COROLLARY 1.1. *If $u, v \in W_p(0, T)$ then $(u(\cdot), v(\cdot))_H$ is absolutely continuous on $[0, T]$ and*

$$\frac{d}{dt}(u(t), v(t))_H = u'(t)(v(t)) + v'(t)(u(t)), \quad \text{a.e.} \quad t \in [0, T].$$

The following compactness criterion will be useful for nonlinear evolution problems.

PROPOSITION 1.3 (LIONS-AUBIN). *Let B_0, B, B_1 be Banach spaces with $B_0 \subset B \subset B_1$; assume $B_0 \hookrightarrow B$ is compact and $B \hookrightarrow B_1$ is continuous. Let $1 < p < \infty$, $1 < q < \infty$, let B_0 and B_1 be reflexive, and define*

$$W \equiv \{u \in L^p(0, T; B_0) : u' \in L^q(0, T; B_1)\}.$$

Then the inclusion $W \hookrightarrow L^p(0, T; B)$ is compact.

LEMMA 1.1. *For each $\delta > 0$ there is a C_δ such that*

$$\|v\|_B \leq \delta\|v\|_{B_0} + C_\delta\|v\|_{B_1}, \quad v \in B_0.$$

PROOF. Otherwise there exists a $\delta > 0$ and a sequence $\{v_n\}$ satisfying $\|v_n\|_{B_0} = 1$ and

$$\|v_n\|_B > \delta\|v_n\|_{B_0} + n\|v_n\|_{B_1}, \quad n \geq 1.$$

The sequence is bounded in B , so $v_n \rightarrow 0$ in B_1 . Also some subsequence $\{v_m\}$ is convergent in B , hence, $v_m \rightarrow 0$ in B , and this shows $v_m \rightarrow 0$ in B_0 , a contradiction. $\square$

PROOF OF PROPOSITION 1.3. Let $\{u_n\}$ be bounded in W . Since W is reflexive there is a subsequence (which we denote again by $\{u_n\}$) such that $u_n \rightharpoonup u$ in W . That is, we have

$$u_n \rightharpoonup u \text{ in } L^p(0, T; B_0), u'_n \rightharpoonup u' \text{ in } L^q(0, T; B_1).$$

We shall show $u_n \rightarrow u$ in $L^p(0, T; B)$. Without loss of generality, we assume $u = 0$.

First we show $u_n(t) \rightarrow 0$ in B_1 for each $t \in [0, T]$. This will be done for $t = 0$, with any other case being similar. Thus we have

$$\begin{aligned} u_n(0) &= u_n(t) - \int_0^t u'_n \\ &= 1/s \int_0^s u_n(t) dt - 1/s \int_0^s \int_0^t u'_n(t) d\tau dt \\ &= 1/s \int_0^s u_n(t) dt - 1/s \int_0^s (s-t) u'_n(t) dt \\ &\equiv a_n + b_n. \end{aligned}$$

Since u'_n is bounded in $L^q(0, T; B_1)$, for an $\varepsilon > 0$ we choose s so small that $\|b_n\|_{B_1} \leq \int_0^s \|u'_n\|_{B_1} \leq \varepsilon/2$. Then, since $u_n \rightharpoonup 0$ in $L^p(0, T; B_0)$, we have $a_n \rightarrow 0$ in B_0 and, hence, $a_n \rightarrow 0$ in B by compactness. Thus $\|a_n\|_{B_1} \leq \varepsilon/2$ for sufficiently large n and we have $u_n(0) \rightarrow 0$ in B_1 .

Second, observe the continuity of $W \hookrightarrow C([0, T], B_1)$ shows that $\{\|u_n\|_{C([0, T], B_1)}\}$ is bounded. By Lebesgue's Theorem 1.4 and the preceding we obtain $u_n \rightarrow 0$ in $L^p(0, T; B_1)$.

Finally, from the Lemma it follows for any $\delta > 0$

$$\begin{aligned} \|u_n\|_{L^p(B)} &\leq \delta \|u_n\|_{L^p(B_0)} + C_\delta \|u_n\|_{L^p(B_1)} \\ &\leq \delta \cdot \sup_m \|u_m\|_{L^p(B_1)} + C_\delta \|u_n\|_{L^p(B_1)}, \end{aligned}$$

and the last term converges to zero, so

$$\lim_{n \rightarrow \infty} \sup \|u_n\|_{L^p(B)} \leq \delta \cdot \sup_m \|u_m\|_{L^p(B_0)}.$$

This shows $u_n \rightarrow 0$ in $L^p(0, T; B)$ as desired. $\square$

III.2. Linear Evolution Equations

We begin our discussion of evolution equations by considering in this and in the following section the case of linear equations. The semigroup approach was given in Section I.5 for the autonomous linear case, and it will be developed for the nonlinear case in Chapter IV. Here we shall consider a time-dependent family of operators and shall resolve the Cauchy problem by a variational method. This is analogous to the development in I.2 for elliptic equations and requires an appropriate extension of the Lax-Milgram Theorem to cover parabolic problems. We shall show with examples that these results are essentially sharp, and they will provide a guide for the nonlinear results which will be developed in the remainder of this Chapter.

We first formulate the *Cauchy problem* or *initial-value problem* for an evolution equation. Let V be a separable Hilbert space with dual V' ; then $\mathcal{V} \equiv L^2(0, T; V)$ is a Hilbert space with dual $\mathcal{V}' = L^2(0, T; V')$. Assume that for each $t \in [0, T]$

we are given a continuous bilinear form $a(t; \cdot, \cdot)$ on V or, equivalently, an operator $\mathcal{A}(t) \in \mathcal{L}(V, V')$,

$$\mathcal{A}(t)u(v) = a(t; u, v) , \quad u, v \in V , \quad t \in [0, T] ,$$

such that for each pair $u, v \in V$ the function $a(\cdot; u, v)$ is in $L^\infty(0, T; \mathbb{R})$. Then by the uniform boundedness principle we obtain a $K > 0$ for which

$$|a(t; u, v)| \leq K \|u\| \|v\| , \quad u, v \in V , \quad t \in [0, T] .$$

Suppose $u \in \mathcal{V}$ and $v \in V$. Then we have $t \mapsto \mathcal{A}(t)u(t)(v) = \mathcal{A}^*(t)v(u(t))$ is measurable, since $t \mapsto \mathcal{A}^*(t)v$ being weakly measurable with V' separable shows it is measurable by Theorem 1.11. It follows likewise that $t \mapsto \mathcal{A}(t)u(t)$ is measurable, hence, by the uniform bound above that it belongs to $\mathcal{V}'$. Note that if $u, v \in \mathcal{V}$ then $t \mapsto a(t; u(t), v(t))$ belongs to $L^1(0, T; \mathbb{R})$. Assume H is a Hilbert space identified with its dual and that the embedding $V \hookrightarrow H$ is dense and continuous; hence, $H \subset V'$ by restriction. Finally, let $u_0 \in H$ and $f \in \mathcal{V}'$ be given.

PROPOSITION 2.1. *The following are equivalent:*

- (a) $u \in \mathcal{V} : u'(t) + \mathcal{A}(t)u(t) = f(t)$ in $\mathcal{V}'$, $u(0) = u_0$.
- (b) $u \in \mathcal{V} : \text{ for every } v \in \mathcal{V} \text{ with } v \in W^{1,2}(0, T; H), v(T) = 0.$

$$- \int_0^T (u(t), v'(t))_H dt + \int_0^T \mathcal{A}(t)u(t)(v(t)) dt = \int_0^T f(t)(v(t)) dt + (u_0, v(0))_H .$$

- (c) $u \in \mathcal{V} : \text{ for each } v \in V$

$$\frac{d}{dt}(u(t), v)_H + \mathcal{A}(t)u(t)(v) = f(t)(v) \quad \text{in } \mathcal{D}^*(0, T)$$

and $u(0) = u_0$.

PROOF. Implicit in (a) is $u \in W_2(0, T)$; thus the initial condition is meaningful by Proposition 1.2. Corollary 1.1 shows that (a) implies (b). If (b) holds, then set $v(t) = \varphi(t)v$ where $v \in V$ and $\varphi \in C_0^\infty(0, T)$ to obtain the equality in $\mathcal{D}^*$ of (c). Then choosing $\varphi \in C^1[0, T]$ with $\varphi(T) = 0$ and setting $v(t)$ as before in (b) we obtain

$$- \int_0^T (u(t), v)_H \varphi'(t) dt - \int_0^T \frac{d}{dt}(u(t), v)_H \varphi(t) dt = (u_0, v) \varphi(0) .$$

That is, $(u(0), v) \varphi(0) = (u_0, v) \varphi(0)$ for all such φ, v , so $u(0) = u_0$ and we have (c). Finally, assume (c). Since $f - \mathcal{A}u \in \mathcal{V}'$ it follows from Proposition 1.1 that $u \in W_2(0, T)$ and (a) holds. $\square$

Note also that the proof shows $u \in C([0, T], H)$ in the case (b) so we had $(u(0), v)_H = \lim_{t \rightarrow 0} (u(t), v)_H$. To find a function u as in Proposition 2.1 will be called the *Cauchy Problem* for the evolution equation given in (a). Also we refer to (a) as the *strong* formulation, to (c) as the *weak* formulation, and to the intermediate case (b) as the *variational formulation*. This is the one we shall first resolve. There are many other intermediate formulations which can be useful.

The following is a refinement of the Lax-Milgram Theorem I.2.2.

THEOREM 2.1 (LIONS). *Let $\{\mathcal{H}, |\cdot|\}$ be a Hilbert space and $\{\Phi, \|\cdot\|\}$ a normed linear space. Suppose $E : \mathcal{H} \times \Phi \rightarrow \mathbb{R}$ is bilinear and that $E(\cdot, \varphi)$ is in $\mathcal{H}'$ for each $\varphi \in \Phi$. Then the following are equivalent:*

$$[C] \quad \inf_{\|\varphi\|=1} \sup_{|u|\leq 1} |E(u, \varphi)| \geq c > 0 ,$$

[E] *for each $f \in \Phi'$ there exists a*

$$u \in \mathcal{H} : E(u, \varphi) = f(\varphi) , \quad \varphi \in \Phi .$$

PROOF. Define the (not necessarily continuous but) linear $K : \Phi \rightarrow \mathcal{H}'$ by $E(u, \varphi) = \langle u, K\varphi \rangle$ for $u \in \mathcal{H}$ and $\varphi \in \Phi$. Condition [C] is precisely the statement that $\|K(\varphi)\|_{\mathcal{H}'} \geq c\|\varphi\|$ for $\varphi \in \Phi$, and this implies that K is invertible, so $K^{-1} : Rg(K) \rightarrow \Phi$ extends by continuity to a linear $\overline{K^{-1}} : \overline{Rg(K)} \rightarrow \hat{\Phi}$, the completion of Φ . Let $f \in \Phi'$; then a $u \in \mathcal{H}$ satisfies $E(u, \varphi) = f(\varphi)$ for $\varphi \in \Phi$ exactly if $\langle u, g \rangle = f(K^{-1}g)$ for all $g \in Rg(K)$. Thus, if $P : \mathcal{H}' \rightarrow \overline{Rg(K)}$ is the orthogonal projection, this would follow from

$$\langle u, g \rangle = f(\overline{K^{-1}}Pg) , \quad g \in \mathcal{H}' .$$

Now $\overline{K^{-1}}P : \mathcal{H}' \rightarrow \hat{\Phi}$ is continuous and has the continuous dual $(\overline{K^{-1}}P)^* : \Phi' = \hat{\Phi}' \rightarrow \mathcal{H}$. Thus we obtain a solution by setting $u \equiv (\overline{K^{-1}}P)^*f$. This shows that [C] implies [E].

Suppose [C] is false. That is, there is a sequence

$$\varphi_n \in \Phi : \|\varphi_n\| = 1 , \quad \sup_{|u|\leq 1} |E(u, \varphi_n)| < 1/n$$

for $n \geq 1$. According to [E], for each $f \in \Phi'$ there is a $u_f \in \mathcal{H}$ such that

$$|f(n\varphi_n)| = |E(u_f, n\varphi_n)| \leq |u_f| , \quad n \geq 1 ,$$

so $\{n\varphi_n\}$ is (weakly) bounded in $\hat{\Phi}$. Thus $\{n\varphi_n\}$ is bounded in Φ by Theorem II.1.2, a contradiction. $\square$

COROLLARY 2.1. *The solution u given above for [E] satisfies the estimate $|u|_{\mathcal{H}} \leq (1/c)\|f\|_{\Phi'}$.*

PROOF. The operator $\overline{K^{-1}}P : \mathcal{H}' \rightarrow \Phi$ has norm equal to $1/c$, and its dual has the same norm. $\square$

COROLLARY 2.2. *Let E_n and f_n be given for each $n \geq 1$ as in Theorem 2.1 with the same constant $c > 0$, and let u_n be the corresponding solutions given in the proof. Then*

$$|u_n - u|_{\mathcal{H}} \leq (1/c)\{\|f_n - f\|_{\Phi'} + \sup_{\|\varphi\|\leq 1} |E_n(u, \varphi) - E(u, \varphi)|\} .$$

PROOF. This follows from the identity

$$E_n(u_n - u, \varphi) = (f_n - f)\varphi - (E_n - E)(u, \varphi) , \quad \varphi \in \Phi ,$$

the linearity of $(\overline{K^{-1}}P)^*$, and Corollary 2.1. $\square$

COROLLARY 2.3. *If Φ is continuously embedded in $\mathcal{H}$, that is, if $|\varphi| \leq c_1 \|\varphi\|$ for all $\varphi \in \Phi$, and if E is Φ -elliptic, that is, there is an $\alpha > 0$ such that*

$$E(\varphi, \varphi) \geq \alpha \|\varphi\|^2, \quad \varphi \in \Phi,$$

then ([C] and hence) Theorem 2.1 holds with $c = \alpha/c_1$.

PROOF. From the elliptic estimate and the definition of K we obtain

$$\alpha \|\varphi\|^2 \leq |\varphi| |K\varphi|_{\mathcal{H}'} \leq c_1 \|\varphi\| |K\varphi|_{\mathcal{H}'},$$

so $(\alpha/c_1) \|\varphi\| \leq |K\varphi|_{\mathcal{H}'}$. □

Let's note the following points. Since $Rg(K)$ is not necessarily dense in $\mathcal{H}'$, much non-uniqueness of solutions u is possible. Moreover we needed only that P be a bounded linear extension of the identity from $\overline{Rg(K)}$ to all of $\mathcal{H}'$. The stability estimate in Corollary 2.1 was shown to hold for *some* solution, not every solution, even if we have the stronger assumption of Φ -coercivity in Corollary 2.3. Finally, the dual operator provided an important part of the proof of Theorem 2.1, so a fully nonlinear version is not immediately obvious.

EXAMPLE 2.A. Here we consider the Cauchy Problem for an ordinary differential equation, namely,

$$u' + u = F \quad \text{on } (0, T), \quad u(0) = u_0.$$

Proposition 2.1 shows we should choose $\mathcal{H} = L^2(0, 1)$, $\Phi = \{\varphi \in H^1(0, T) : \varphi(T) = 0\}$, $f(\varphi) = \int_0^T F(t)\varphi(t) dt + u_0 \cdot \varphi(0)$, and $E(u, \varphi) = \int_0^T u(\varphi - \varphi') dt$. Note in the proof of Theorem 2.1 we have $K(\varphi) = \varphi - \varphi' \in \mathcal{H} = \mathcal{H}'$ for $\varphi \in \Phi$. Also we find

$$E(\varphi, \varphi) = \int_0^T |\varphi|^2 dt + (1/2)|\varphi(0)|^2, \quad \varphi \in \Phi,$$

so $E(\cdot, \cdot)$ is $L^2 = \mathcal{H}$ coercive but not H^1 coercive. However we do have

$$|K(\varphi)|_{\mathcal{H}}^2 = \int_0^T (\varphi - \varphi')^2 dt = \|\varphi\|_{H^1(0, T)}^2 + |\varphi(0)|^2,$$

so condition [C] holds with the (much stronger) H^1 norm on Φ , and Theorem 2.1 applies to the case of a rather large dual space $\Phi' \supset C[0, T]'$, since $\Phi \subset C[0, T]$ when $\|\varphi\|^2 = \|\varphi\|_{H^1}^2 + |\varphi(0)|^2$. On the other hand, we can apply Corollary 2.3 when the norm on Φ is no stronger than $\|\varphi\|^2 = \|\varphi\|_{L^2}^2 + |\varphi(0)|^2$ and, hence, we have a smaller $\Phi' \subset L^2(0, T) \times \mathbb{R}$. This example illustrates that Corollary 2.3 is a rather special case of Theorem 2.1, and it is usually quite easy to verify its hypotheses.

EXAMPLE 2.B. We consider next an *initial-boundary-value problem* on a *non-cylindrical* domain. Let $G \subset \mathbb{R}^2$ be a bounded domain whose boundary ∂G is a smooth curve with unit outward normal $n = (n_x, n_t)$ at each point, and consider the linear parabolic heat equation

$$u_t + \lambda u - u_{xx} = F, \quad (x, t) \in G$$

with $u = 0$ on the non-horizontal sides of G and $u = u_0$ on the horizontal bottom of G . We choose spaces

$$\mathcal{H} = \{u \in L^2(G) : \partial_x u \in L^2(G) \text{ and } (un_x)|_{\partial G} = 0\},$$

$$\Phi = \{\varphi \in \mathcal{H} \cap H^1(G) : \varphi|_{\partial G} = 0 \text{ where } n_t = 1\},$$

and the bilinear form

$$E(u, \varphi) = \int_G (\partial_x u \partial_x \varphi + \lambda uv - u \partial_t \varphi), \quad u \in \mathcal{H}, \varphi \in \Phi.$$

For $\varphi \in \Phi$ we compute

$$E(\varphi, \varphi) = \int_G ((\partial_x \varphi)^2 + \lambda \varphi^2) + 1/2 \int_B \varphi^2$$

where B is that portion of ∂G where $n_t = -1$, the *bottom* of G . By definition, the first term is just $|\varphi|_{\mathcal{H}}^2$ and the entire expression is $\|\varphi\|_{\Phi}^2$; note that $\mathcal{H}$ is complete and that $\Phi' = \mathcal{H}' \oplus L^2(B)$. Thus, if $F \in L^2(G)$ and $u_0 \in L^2(B)$ are given, then

$$f(\varphi) \equiv \int_G F \varphi + \int_B u_0 \varphi, \quad \varphi \in \Phi,$$

defines $f \in \Phi'$, and the problem above for the heat equation is in the form of Corollary 2.3. Note that [C] holds with a much stronger norm on Φ , just as in the preceding example.

The corresponding non-homogeneous case can be reduced to the above in this situation. If $U \in H^1(G)$ and if u is a solution of the heat equation with $u = U$ on those points of ∂G where $n_t < 1$, then $v \equiv u - U$ is a solution of

$$v_t + \lambda v - v_{xx} = F - (U_t + \lambda U - U_{xx})$$

with $v = 0$ on the sides and bottom of G , and conversely. In particular, we may choose such a U with $U_{xx} = 0$ and then the prescribed boundary values need only be smooth enough to give $U_t + \lambda U \in L^2(G)$.

Finally we return to the *abstract Cauchy problem*

$$(2.1) \quad u \in L^2(0, T; V) : u' + \mathcal{A}u = f \text{ in } L^2(0, T; V'), \quad u(0) = u_0$$

where the separable Hilbert spaces $V \hookrightarrow H \hookrightarrow V'$, bounded and measurable operators $\mathcal{A}(t) : V \rightarrow V'$, and $f \in L^2(0, T; V')$, $u_0 \in H$ are given as before. If u is a solution of the Cauchy problem then by Proposition 1.2 we apply the equation to u and obtain by integrating

$$(1/2)|u(s)|_H^2 + \int_0^s \mathcal{A}(t)u(t)(u(t)) dt = \int_0^s f(t)(u(t)) dt + 1/2|u_0|_H^2, \quad 0 \leq s \leq T.$$

The usual linearity argument gives the following *uniqueness* result.

PROPOSITION 2.2. *If each $\mathcal{A}(t)$ is monotone, i.e.,*

$$\mathcal{A}(t)v(v) \geq 0, \quad v \in V, \quad t \in [0, T],$$

then there is at most one solution to the Cauchy problem.

To obtain existence of a solution we set $\mathcal{H} = L^2(0, T; V)$ with the usual scalar-product and $\Phi = \{\varphi \in \mathcal{H} : \varphi' \in L^2(0, T; H), \varphi(T) = 0\}$ with the norm given by $\|\varphi\|^2 = |\varphi|_{\mathcal{H}}^2 + |\varphi(0)|_H^2$. By Proposition 2.1 the Cauchy problem is precisely

$$u \in \mathcal{H} : E(u, \varphi) = f(\varphi) , \quad \varphi \in \Phi ,$$

where we define

$$E(u, \varphi) = \int_0^T \{\mathcal{A}(t)u(t)(\varphi(t)) - (u(t), \varphi'(t))_H\} dt$$

and

$$f(\varphi) = \int_0^T f(t)(\varphi(t)) dt + (u_0, \varphi(0))_H .$$

From Corollary 2.3 we obtain an easy sufficient condition for existence.

PROPOSITION 2.3. *Assume the operators are uniformly coercive: there is a $c > 0$ such that*

$$\mathcal{A}(t)v(v) \geq c\|v\|_V^2 , \quad v \in V , \quad t \in [0, T] ,$$

Then there exists a unique solution of the Cauchy problem, and it satisfies

$$(2.2) \quad \|u\|_{L^2(0, T; V)}^2 \leq (1/c)^2 (\|f\|_{L^2(0, T; V')}^2 + |u_0|_H^2) .$$

The preceding result can be extended in two directions. First, $u \in \mathcal{H}$ if and only if $v \in \mathcal{H}$ where $v(t) \equiv e^{-\lambda t}u(t)$, $0 \leq t \leq T$, and u is a solution of the preceding Cauchy problem exactly when v is the corresponding solution of the problem

$$v \in \mathcal{H} : v' + (\mathcal{A}(\cdot) + \lambda I)v = e^{-\lambda t}f(t) , \quad v(0) = u_0 .$$

(This useful device is known as the *exponential shift*.) Thus a sufficient condition for existence by Proposition 2.3 is that there exist a $\lambda \in \mathbb{R}$ and $c > 0$ such that

$$\mathcal{A}(t)v(v) + \lambda|v|_H^2 \geq c\|v\|_V^2 , \quad v \in V , \quad t \in [0, T] .$$

Similarly uniqueness is obtained from such an estimate, even with $c = 0$.

Finally, we consider in more detail the situation of Proposition 2.3; here we identify the operator $K : \Phi \rightarrow \mathcal{H}'$ from the proof of Theorem 2.1 as $K(\varphi) = \mathcal{A}^*(t)\varphi(t) - \varphi'(t)$. If each $\mathcal{A}(t)$ is self-adjoint ($\mathcal{A}(t)^* = \mathcal{A}(t)$) then we can renorm $\mathcal{H}$ by

$$|u|_{\mathcal{H}} = \int_0^T \mathcal{A}(t)u(t)(u(t)) dt .$$

The dual norm on $f \in \mathcal{H}'$ is given by

$$\|f\|_{\mathcal{H}'} = \int_0^T f(t)(\mathcal{A}(t)^{-1}f(t)) dt ,$$

and for those $\varphi \in \Phi$ we have

$$\begin{aligned} \|K(\varphi)\|_{\mathcal{H}'} &= \int_0^T \langle \mathcal{A}(t)\varphi(t) - \varphi'(t) , \varphi(t) - \mathcal{A}(t)^{-1}\varphi'(t) \rangle dt \\ &= \int_0^T \{\mathcal{A}(t)\varphi(t)(\varphi(t)) + \varphi'(t)(\mathcal{A}(t)^{-1}\varphi'(t))\} dt + |\varphi(0)|_H^2 \\ &= \|\varphi\|_{\mathcal{H}}^2 + \|\varphi'\|_{\mathcal{H}'}^2 + |\varphi(0)|_H^2 . \end{aligned}$$

Thus we can apply Theorem 2.1 when Φ has the much stronger norm given by

$$\|\varphi\|_{\Phi}^2 = \|\varphi\|_{L^2(0,T;V)}^2 + \|\varphi'\|_{L^2(0,T;V')}^2 = \|\varphi\|_{W_2(0,T)}^2 .$$

In particular, this permits a much larger class of data $f \in \Phi'$ for which the Cauchy problem can be solved.

With the same assumptions as in Proposition 2.3 we can also uniquely solve the *Periodic Problem*: find

$$u \in \mathcal{V} : u'(t) + \mathcal{A}(t)u(t) = f(t) \text{ in } \mathcal{V}' , \quad u(0) = u(T) .$$

More generally we have the following.

PROPOSITION 2.4. *In the situation of Proposition 2.3 and with a given linear contraction $B : H \rightarrow H$, there is a unique*

$$(2.3) \quad u \in \mathcal{V} : u'(t) + \mathcal{A}(t)u(t) = f(t) \text{ in } \mathcal{V}' , \quad -B^*u(T) + u(0) = u_0 ,$$

where B^* is the adjoint of B , and it is characterized by

$$u \in \mathcal{V} : \text{ for every } v \in \mathcal{V} \text{ with } v \in W^{1,2}(0,T;H) \text{ and } v(T) = Bv(0) \\ - \int_0^T (u(t), v'(t))_H dt + \int_0^T \mathcal{A}(t)u(t)(v(t)) dt = \int_0^T f(t)(v(t)) dt + (u_0, v(0))_H .$$

Here's another application of Theorem 2.1, and this yields a *stronger* or more *regular* notion of a solution in the situation of Proposition I.4.1.

PROPOSITION 2.5. *Let the self-adjoint operator A be constructed from a symmetric non-negative form as in Proposition I.4.1, $f \in L^2(0,T;H)$ and $u_0 \in V$. Then there is a unique*

$$(2.4) \quad u \in W^{1,2}(0,T;H) : u' + Au = f \text{ in } L^2(0,T;H) , \quad u(0) = u_0 .$$

Specifically, we have $u(t) \in D(A)$ at a.e. $t \in (0,T)$.

PROOF. By the change of variable $u(t) = e^t \tilde{u}(t)$ we obtain an equivalent problem with A replaced by $A + I$, so we may assume A^{-1} is a contraction on H . Now define

$$\mathcal{H} = L^2(0,T;D(A)) = \{u \in L^2(0,T;H) : Au(\cdot) \in L^2(0,T;H)\}$$

with the norm given by $|u|_{\mathcal{H}} = \|Au\|_{L^2(0,T;H)}$ and set $\Phi = \{\varphi \in \mathcal{H} : A\varphi \in W^{1,2}(0,T;H) , A\varphi(T) = 0\}$. With the bilinear form

$$E(u, \varphi) = \int_0^T \left\{ (Au(t), A\varphi(t))_H - (u(t), \frac{d}{dt} A\varphi(t))_H \right\} dt$$

and linear functional

$$f(\varphi) = \int_0^T (f(t), A\varphi(t))_H + (u_0, A\varphi(0))_H , \quad \varphi \in \Phi$$

we find

$$E(\varphi, \varphi) = |\varphi|_{\mathcal{H}}^2 + 1/2 (A\varphi(0), \varphi(0))_H .$$

By our assumption on A we see that $(\cdot, A\cdot)_H$ is the scalar-product on V , so we are in the situation of Corollary 2.3 with $E(\varphi, \varphi) = \|\varphi\|_{\Phi}^2$. Finally, from

$$u \in \mathcal{H} : E(u, \varphi) = f(\varphi) , \quad \varphi \in \Phi ,$$

it remains to check that $u \in W^{1,2}(0, T; H)$ by Proposition 1.1 and then that u is a solution. $\square$

Since the solution u in Proposition 2.5 satisfies $u \in L^2(0, T; D(A))$, and $\dot{u} \in L^2(0, T; H)$, it follows as in the proof of Proposition 1.2 that $u \in C([0, T], V)$. (For this we use the equivalence of $(Au, v)_H$ with the V scalar-product, $(u, v)_V$.) Thus the hypothesis $u_0 \in V$ is optimal for this class of solution.

Let's show that the Cauchy problem can be solved with initial data in H for an appropriately generalized notion of solution. By linearity it suffices to consider the homogeneous equation, so let u be the solution of

$$u \in W^{1,2}(0, T; H) : u' + Au = 0, u(0) = u_0$$

with $u_0 \in V$. Take the scalar-product of the equation in H with u and integrate to obtain the *energy estimate*

$$\frac{1}{2}|u(t)|_H^2 + \int_0^t (Au(s), u(s))_H ds \leq \frac{1}{2}|u_0|_H^2.$$

Next take the scalar-product with $tu'(t)$ and integrate to obtain

$$\int_0^t |u'(s)|_H^2 ds + \frac{T}{2}(Au(T), u(T))_H \leq \frac{1}{2} \int_0^T (Au(s), u(s))_H ds.$$

From these two estimates follows

$$(2.5) \quad 4 \int_0^T |u'(t)|_H^2 dt + 2T(Au(T), u(T))_H + |u(T)|_H^2 \leq |u_0|_H^2.$$

Thus, $\|t^{1/2}u'\|_{L^2(0, T; H)}$, $\|u\|_{C([0, T], H)}$ and $\|u\|_{C([\delta, T], V)}$ for any $\delta > 0$ are bounded by $|u_0|_H$. To apply this estimate, let $\{u_0^n\}$ be a sequence in V which converges (in H -norm) to $u_0 \in H$. Let $u_n(t)$ be the solution with $u_n(0) = u_0^n$. From the last estimate applied to differences, we obtain Cauchy-type estimates which show convergence of $\{t^{1/2}u'_n\}$ in $L^2(0, T; H)$, $\{u_n\}$ in $C([0, T], H)$ and in $C([\delta, T], V)$ for every $\delta, 0 < \delta < T$. This proves the following *regularity theorem* for the situation of Proposition 2.3.

COROLLARY 2.4. *Let A and f be given as in Proposition 2.5, and let $u_0 \in H$. The unique solution u of the Cauchy problem (2.1) satisfies $t^{1/2}u' \in L^2(0, T; H)$, (2.5), and, for each $\delta > 0$, $u \in W^{1,2}(\delta, T; H)$, $u \in C([\delta, T], V)$.*

See Example 3.B below for a typical application to initial-boundary-value problems for parabolic equations.

III.3. Linear Degenerate Equations

Let the separable Hilbert spaces V and $\mathcal{V} = L^2(0, T; V)$ with duals V' and $\mathcal{V}' = L^2(0, T; V')$ be given. Suppose also that W is a Hilbert space containing V and the injection $V \hookrightarrow W$ is continuous with V dense in W ; thus the restriction map $W' \hookrightarrow V'$ is a continuous injection. Assume that for each $t \in [0, T]$ we are given an operator $\mathcal{A}(t) \in \mathcal{L}(V, V')$ such that $\mathcal{A}(\cdot)u(v) \in L^\infty(0, T)$ for each pair $u, v \in V$, as in Section 2, and likewise we are given another family of operators $\mathcal{B}(t) \in \mathcal{L}(W, W')$

with $\mathcal{B}(\cdot)u(v) \in L^\infty(0, T)$ for each pair $u, v \in W$. Finally, suppose $u_0 \in W$ and $f \in L^2(0, T; V')$ are given. The *Cauchy Problem* considered here is to find

$$u \in \mathcal{V} : \text{ for all } v \in \mathcal{V} \cap W^{1,2}(0, T; W) \quad \text{with} \quad v(T) = 0 \\ - \int_0^T \mathcal{B}(t)u(t)(v'(t)) dt + \int_0^T \mathcal{A}(t)u(t)(v(t)) dt = \int_0^T f(t)(v(t)) dt + \mathcal{B}(0)u_0(v(0)) .$$

Just as in Proposition 2.1 we can show that this is equivalent to

$$(3.1) \quad u \in \mathcal{V} : \frac{d}{dt}(\mathcal{B}(t)u(t)) + \mathcal{A}(t)u(t) = f(t) \quad \text{in } \mathcal{V}' , \quad (\mathcal{B}u)(0) = \mathcal{B}(0)u_0 ,$$

where the initial condition has meaning since $t \mapsto \mathcal{B}(t)u(t)$ is absolutely continuous in V' .

We shall give sufficient conditions for the Cauchy Problem (3.1) to be well-posed; existence will be another application of Theorem 2.1.

DEFINITION. The family $\{\mathcal{A}(t) : t \in [0, T]\}$ of operators given as above will be called *regular* if for each pair $u, v \in V$ the function $\mathcal{A}(\cdot)u(v)$ is absolutely continuous on $[0, T]$ and there is a $K(\cdot) \in L^1(0, T)$ such that

$$\left| \frac{d}{dt} \mathcal{A}(t)u(v) \right| \leq K(t) \|u\| \|v\| , \quad u, v \in V , \quad \text{a.e. } t \in [0, T] .$$

PROPOSITION 3.1. *If $\{\mathcal{A}(t) : 0 \leq t \leq T\}$ is a regular family of operators in $\mathcal{L}(V, V')$, then for each pair $u, v \in W^{1,2}(0, T; V)$ the function $\mathcal{A}(\cdot)u(\cdot)(v(\cdot))$ is absolutely continuous on $[0, T]$ and its derivative is given by*

$$\begin{aligned} & \frac{d}{dt} \mathcal{A}(t)u(t)(v(t)) \\ &= \mathcal{A}(t)u'(t)(v(t)) + \mathcal{A}(t)u(t)(v'(t)) + \mathcal{A}'(t)u(t)(v(t)) , \quad \text{a.e. } t \in [0, T] . \end{aligned}$$

PROOF. Fix $u \in V$ and let $\{v_n : n \geq 1\}$ be dense in V . For each n define $\mathcal{A}'(t)u(v_n) = \frac{d}{dt} \mathcal{A}(t)u(v_n)$ at a.e. $t \in [0, T]$ and then note by the estimate above that this determines $\mathcal{A}'(t)u$ a.e. on $[0, T]$. This function is weakly, hence, strongly measurable and the above estimate shows it belongs to $L^1(0, T; V')$. The weak absolute continuity of $\mathcal{A}(\cdot)u$ then shows that $\mathcal{A}(t)u = \mathcal{A}(0)u + \int_0^t \mathcal{A}'(s)u ds$ by Proposition 1.1., so $\mathcal{A}(\cdot)u \in W^{1,1}(0, T; V')$. Next let $u \in \mathcal{V}$. For each $v \in V$ we see that $\mathcal{A}(t)u(t)(v) = \mathcal{A}^*(t)v(u(t))$ is absolutely continuous, since the above shows the function $\mathcal{A}^*(t)v$ is absolutely continuous, and thus we may integrate the (strong) derivative $\frac{d}{dt}(\mathcal{A}(t)u(t)) = \mathcal{A}'(t)u(t) + \mathcal{A}(t)u'(t) \in L^1(0, T; V')$. The desired result now follows easily. $\square$

Rather general sufficient conditions for existence are given as follows.

PROPOSITION 3.2. *Let the separable Hilbert spaces V, W and linear operators $\mathcal{A}(t), \mathcal{B}(t), 0 \leq t \leq T$, the data $u_0 \in W$ and $f \in L^2(0, T; V')$ be given as above, and assume further that $\mathcal{B}(t)$ is a regular family of Hermitian (self-adjoint) operators. In addition, suppose $\mathcal{B}(0)$ is monotone and there are numbers λ and $c > 0$ such that*

$$(3.2) \quad 2\mathcal{A}(t)v(v) + \lambda\mathcal{B}(t)v(v) + \mathcal{B}'(t)v(v) \geq c\|v\|^2 , \quad v \in V , \quad 0 \leq t \leq T .$$

Then there exists a solution u of the Cauchy Problem (3.1), and it satisfies

$$\|u\|_{L^2(0,T;V)} \leq C(\lambda, c) \left(\|f\|_{L^2(0,T;V')}^2 + \mathcal{B}(0)u_0(u_0) \right)^{1/2}.$$

PROOF. By the exponential shift we may assume $\lambda = 0$. Define $\mathcal{H} = L^2(0, T; V)$ with the usual norm and $\Phi = \{\varphi \in \mathcal{H} : \varphi' \in L^2(0, T; W), \varphi(T) = 0\}$ with norm given by $\|\varphi\|_\Phi^2 = \|\varphi\|_\mathcal{H}^2 + \mathcal{B}(0)\varphi(0)(\varphi(0))$. For $u \in \mathcal{H}$ and $\varphi \in \Phi$ set

$$\begin{aligned} E(u, \varphi) &= \int_0^T (\mathcal{A}(t)u(t)(\varphi(t)) - \mathcal{B}(t)u(t)(\varphi'(t))) dt, \\ f(\varphi) &= \int_0^T f(t)\varphi(t) dt + \mathcal{B}(0)u_0(\varphi(0)). \end{aligned}$$

Then we use Proposition 3.1 to check that

$$\begin{aligned} 2E(\varphi, \varphi) &= \int_0^T 2\mathcal{A}(t)\varphi(t)(\varphi(t)) dt + \int_0^T \left\{ \mathcal{B}'(t)\varphi(t)(\varphi(t)) - \frac{d}{dt}\mathcal{B}(t)\varphi(t)(\varphi(t)) \right\} dt \\ &\geq c\|\varphi\|_\mathcal{H}^2 + \mathcal{B}(0)\varphi(0)(\varphi(0)), \end{aligned}$$

so Corollary 2.3 applies. $\square$

COROLLARY 3.1. *If there are numbers λ , α and $c > 0$ such that*

$$\begin{aligned} \mathcal{B}'(t)v(v) + \alpha\mathcal{B}(t)v(v) &\geq 0, \\ \mathcal{A}(t)v(v) + \lambda\mathcal{B}(t)v(v) &\geq c\|v\|^2, \quad v \in V, \quad \text{a.e. } t \in [0, T], \end{aligned}$$

then the coercive estimate above and, hence, Proposition 3.2 are valid. In this case,

$$\mathcal{B}(t)v(v) \geq \exp(-\lambda t)\mathcal{B}(0)v(v), \quad v \in V, \quad 0 \leq t \leq T$$

so each $\mathcal{B}(t)$ is monotone.

Uniqueness of solutions to (3.1) can be shown when the second family of operator $\{\mathcal{A}(t)\}$ is regular.

PROPOSITION 3.3. *Let the separable Hilbert spaces V , W and linear operators $\mathcal{A}(t)$, $\mathcal{B}(t)$, $0 \leq t \leq T$, the data $u_0 \in W$ and $f \in L^2(0, T; V')$ be given for the Cauchy Problem (3.1). In addition, assume each $\mathcal{B}(t)$ is monotone, $\mathcal{A}(t)$ is a regular family of Hermitian operators, and there is a pair of numbers λ , $c > 0$ such that*

$$(3.3) \quad \mathcal{A}(t)v(v) + \lambda\mathcal{B}(t)v(v) \geq c\|v\|^2, \quad v \in V, \quad \text{a.e. } t \in [0, T].$$

Then there is at most one solution of the Cauchy Problem (3.1).

PROOF. Let u be a solution with $u_0 = 0$ and $f = 0$. The difficulty is that u is not in Φ , not necessarily even differentiable, so we construct a test function as follows. Let $s \in (0, T)$ and define $v(t) = -\int_t^s u$ for $0 \leq t \leq s$ and $v(t) = 0$ for $s \leq t \leq T$. Then $v \in W^{1,2}(0, T; V)$, $v'(t) = u(t)$ for $0 < t < s$ and $v(t) = v'(t) = 0$ for $s < t \leq 1$. Since $v \in \Phi$ we have

$$-\int_0^s \mathcal{B}(t)u(t)(u(t)) dt + \int_0^s \mathcal{A}(t)v'(t)(v(t)) dt = 0$$

and then from Proposition 3.1 there follows

$$\int_0^s \{2\mathcal{B}(t)u(t)(u(t)) + \mathcal{A}'(t)v(t)(v(t))\} dt + \mathcal{A}(0)v(0)(v(0)) = 0 .$$

As before we may assume $\lambda = 0$ in our hypothesis. Set $U(t) = \int_0^t u$ so that $U(s) = -v(0)$ and $U(t) - U(s) = v(t)$ for $t \in (0, s)$. Then we have

$$\begin{aligned} c\|U(s)\|^2 &\leq \mathcal{A}(0)U(s)(U(s)) + \int_0^s 2\mathcal{B}(t)u(t)(u(t)) dt \\ &= - \int_0^s \mathcal{A}'(t)v(t)(v(t)) dt \leq \int_0^s K(t)\|v(t)\|^2 dt \\ &\leq 2 \int_0^s K(t) \left(\|U(t)\|^2 + \|U(s)\|^2 \right) dt . \end{aligned}$$

Choose s_0 so small that $k \equiv 2 \int_0^{s_0} K < c$. Then for $0 \leq s \leq s_0$,

$$\|U(s)\|^2 \leq 2/(c - k) \int_0^s K(t)\|U(t)\|^2 dt ,$$

and this implies that $U(s) = 0$ for $0 \leq s \leq s_0$. Thus, $u(t) = 0$ on $(0, s_0)$. Since $K(\cdot)$ is integrable we could find an s_0 such that $\int_\tau^{\tau+s} K < c/2$ for all $0 \leq \tau \leq T - s_0$ and apply the preceding a finite number of times to obtain $u = 0$ on all of $[0, T]$. $\square$

We shall illustrate the preceding results with two general examples. These are mixed *elliptic-parabolic* partial differential equations for which we find appropriate well-posed problems.

EXAMPLE 3.A. Let H^1 denote the Sobolev space $\{v \in L^2(0, 1) : \partial v \in L^2(0, 1)\}$ with the usual norm, $\|v\|_{H^1}^2 = \|v\|_{L^2}^2 + \|\partial v\|_{L^2}^2$, and denote by H_0^1 the subspace consisting of those members which vanish at the end points. From the calculus of variations comes the estimate

$$\pi\|v\|_{L^2} \leq \|\partial v\|_{L^2} , \quad v \in H_0^1 .$$

Choose $V = H_0^1$ and recall V' is a space of distributions on the unit interval. For our second space, choose $W = \{v \in H^1 : v(1) = 0\}$. If $v \in W$ we have

$$|v(x)| = \left| \int_x^1 \partial v \right| \leq |x - 1|^{1/2} \|\partial v\|_{L^2} , \quad 0 \leq x \leq 1 ,$$

and thus the estimate

$$\sup \{|v(x)| : 0 \leq x \leq 1\} \leq \|\partial v\|_{L^2} , \quad v \in W .$$

Let β be an absolutely continuous function from $[0, T]$ into the unit interval and define

$$\mathcal{B}(t)u(v) \equiv \int_0^{\beta(t)} u(x)v(x) dx , \quad u, v \in W , \quad 0 \leq t \leq T .$$

Since the integrand is bounded this is differentiable and we have

$$\mathcal{B}'(t)u(v) = u(\beta(t))v(\beta(t))\beta'(t) \quad \text{a.e. } t \in [0, T] ,$$

and from above we obtain

$$|\mathcal{B}'(t)u(v)| \leq \|\partial u\|_{L^2} \|\partial v\|_{L^2} |\beta'(t)|, \quad \text{a.e. } t \in [0, T],$$

so $\{\mathcal{B}(t)\}$ is a regular family of Hermitian monotone operators on W . Similarly define a family of operators on V by

$$\mathcal{A}(t)u(v) = \int_0^1 \partial u \partial v \, dx, \quad u, v \in V, \quad 0 \leq t \leq T.$$

These satisfy the assumptions in Proposition 3.3 with $\lambda = 0$, so for a given $u_0 \in W$ and $F \in L^2(0, 1) \times (0, T)$ we define $f \in L^2(0, T; V')$ by

$$f(t)(v) = \int_0^1 F(x, t)v(x) \, dx, \quad 0 \leq t \leq 1$$

and obtain the uniqueness of weak solutions of the initial-boundary-value problem

$$(3.4) \quad \begin{aligned} \partial_t u - \partial_x^2 u &= F(x, t), & 0 < x < \beta(t), \\ -\partial_x^2 u &= F(x, t), & \beta(t) < x < 1, \\ u(0, t) &= u(1, t) = 0, & 0 < t < T, \\ u(x, 0) &= u_0(x), & 0 < x < \beta(0). \end{aligned}$$

Note that we have a parabolic (elliptic) equation on the left (respectively, right) side of the graph of β and that the initial-condition is only in the parabolic region. Furthermore, suppose there is a number σ , $0 < \sigma < 2$, such that

$$\beta'(t) \geq \sigma - 2, \quad \text{a.e. } t \in [0, T].$$

Then from above follows

$$2\mathcal{A}(t)v(v) + \mathcal{B}'(t)v(v) \geq \sigma \|\partial v\|_{L^2}^2, \quad v \in V,$$

so we obtain (3.2) and existence of a weak solution from Proposition 3.2. Note that Corollary 3.1 does not work here. Also, the curve $x = \beta(t)$ is non-characteristic at a.e. t , but it may take the characteristic direction at certain points, e.g., $(x - 1/2)^3 = (t - T/2)/4T$.

In the preceding example it was essential to have the L^∞ bounds which occur in the one-dimensional case. These were used not only to get coercivity from the lower-bound on β' but also to show that $\{\mathcal{B}(t)\}$ is a regular family. The following elementary result is rather useful in the construction of regular families on L^2 in any dimension.

LEMMA 3.1. *Let $F : G \times (0, T) \rightarrow \mathbb{R}$ be measurable and assume $F(\cdot, t) \in L^1(G)$ for $t \in [0, T]$, $F(x, \cdot)$ is absolutely continuous for a.e. $x \in G$, and $\partial_t F \in L^1(G \times (0, T))$. Then the function $t \mapsto \int_G F(x, t) \, dx$ is absolutely continuous and*

$$\frac{d}{dt} \int_G F(x, t) \, dx = \int_G \partial_t F(x, t) \, dx.$$

PROOF. From the Fubini Theorem we have

$$\int_0^t \left\{ \int_G \partial_t F(x, t) dx \right\} dt = \int_G \left\{ \int_0^t \partial_t F(x, t) dt \right\} dx = \int_G (F(x, t) - F(x, 0)) dx$$

and the result follows immediately. $\square$

COROLLARY 3.2. Suppose $b \in L^\infty(G \times (0, T))$, $b(x, \cdot)$ is absolutely continuous for a.e. $x \in G$ and $|\partial_t b(x, t)| \leq K(t)$ for a.e. $t \in [0, T]$ with $K \in L^1(0, T)$. Then the operators

$$\mathcal{B}(t)u(v) \equiv \int_G b(x, t)u(x)v(x) dx, \quad u, v \in L^2(G)$$

are a regular family on $L^2(G)$.

PROOF. Set $F(x, t) = b(x, t)u(x)v(x)$ in Lemma 3.1. $\square$

We give another illustration of our abstract results in one spatial dimension. This is merely for simplicity since similar constructions are straightforward in higher dimension.

EXAMPLE 3.B. Choose $V = H_0^1$ and $\{\mathcal{A}(t)\}$ just as in Example 1, but set $W = L^2(0, 1)$ and define $\mathcal{B}(t)$ as in Corollary 3.2 where the function $b(\cdot)$ is described with $G = (0, 1)$. Let $u_0 \in L^2$ and let F, f be given as in Example 3.A. Then our Cauchy Problem corresponds to generalized solutions of the initial-boundary-value problem

$$(3.5) \quad \begin{aligned} \partial_t (b(x, t)u(x, t)) - \partial_x^2 u(x, t) &= F(x, t), \\ u(0, t) = u(1, t) &= 0, \quad 0 < t < T, \\ b(x, 0)u(x, 0) &= b(x, 0)u_0(x), \quad 0 < x < T. \end{aligned}$$

We consider separately the questions of uniqueness and existence. Note that both families of operators are regular and Hermitian and that $\mathcal{A}(t)$ is V -coercive. Thus Proposition 3.3 applies to give uniqueness if (and only if) the $\mathcal{B}(t)$ are monotone, i.e., $b(x, t) \geq 0$ a.e. $(x, t) \in (0, 1) \times (0, T)$. In order to apply Proposition 3.2 it is sufficient to assume there is a σ , $0 < \sigma < 2\pi^2$ such that

$$(3.6) \quad b(x, 0) \geq 0 \text{ and } \partial_t b(x, t) \geq (\sigma - 2\pi^2), \quad \text{a.e. } (x, t) \in (0, 1) \times (0, T).$$

Note that this implies

$$\mathcal{B}'(t)v(v) \geq (\sigma - 2\pi^2)\|v\|_{L^2}^2, \quad v \in L^2,$$

so the required estimate in Proposition 3.2 holds with $\lambda = 0$; here we have used the first estimate in Example 3.A.

Note that the conditions (3.6) and that $b(\cdot, \cdot)$ be non-negative are independent: neither implies the other. Specifically (3.6) permits $b(x, t)$ to attain negative values over large regions, and there the partial differential equation is *backward parabolic*. This occurs in the equation

$$\partial_t \{(\pi^2/4)(1 - 2t)^3 u(x, t)\} = \partial_x^2 u(x, t)$$

where we have $b'(t) = -(3\pi^2/2)(1 - 2t)^2 \geq -3\pi^2/2$ so (3.6) holds and there exists a solution for $0 \leq t \leq T$. A solution with $u_0(x) = \sin(\pi x)$ is given by $u(x, t) =$

$(1-2t)^{-3} \exp(1-(1-2t)^{-2}) \sin(\pi x)$ and a second is obtained changing the values to zero for all $t \in (1/2, 1]$. Proposition 3.3 holds only on $0 \leq t \leq 1/2$ where $b(x, t) \geq 0$, so it is sharp in this example.

Finally let $n \geq 1$ and consider the Cauchy Problem for the equation

$$\partial_t((\pi^2/n)(1-t)^{1/2}u(x, t)) = \partial_x^2 u(x, t)$$

and the initial value u_0 given before. Since $b(x, t) \geq 0$ there is at most one solution on (any subinterval of) $0 \leq t \leq 1$; however (3.6) holds only on $[0, T]$ with $T < 1 - (4n)^{-2}$ and Proposition 3.2 asserts existence of a solution on this interval. This solution is given by

$$u(x, t) = (1-t)^{-1/2} \exp(2n(1-t)^{1/2}) \sin(\pi x) ,$$

and it is clearly *not* in $L^2((0, 1) \times (0, 1))$, so there does not exist a solution on $0 \leq t \leq 1$. This example shows Proposition 3.2 is quite sharp.

III.4. Nonlinear Parabolic Equations

Let V be a reflexive Banach space with dual V' ; we shall set $\mathcal{V} = L^p(0, T; V)$ for $1 < p < \infty$, and its dual is $\mathcal{V}' \cong L^{p'}(0, T; V')$. Also we have a Hilbert space H which will be identified with its dual by the Riesz map, and we assume V is dense and continuously embedded in H . Thus we are in the situation of Proposition 1.2. Suppose we are given a (not necessarily linear) function $\mathcal{A} : V \rightarrow V'$ and $u_0 \in H$, $f \in \mathcal{V}'$. Then consider the *Cauchy Problem* to find

$$(4.1) \quad u \in \mathcal{V} : u'(t) + \mathcal{A}(u(t)) = f(t) \quad \text{in } \mathcal{V}' , \quad u(0) = u_0 \text{ in } H .$$

It is understood that $u' \in \mathcal{V}'$ in (4.1), so u is continuous into H and the condition on $u(0)$ is meaningful. If $\mathcal{A}$ is known to map $\mathcal{V}$ into $\mathcal{V}'$, i.e., the realization of $\mathcal{A} : V \rightarrow V'$ as an operator on $\mathcal{V}$ has values in $\mathcal{V}'$, then (4.1) is equivalent to

$$u \in \mathcal{V} : \text{for every } v \in \mathcal{V} \text{ with } v' \in \mathcal{V}' \text{ and } v(T) = 0 \\ - \int_0^T v'(t)(u(t)) dt + \int_0^T \mathcal{A}(u(t))v(t) dt = \int_0^T f(t)v(t) dt + (u_0, v(0))_H .$$

There is a corresponding statement “weakly” in $\mathcal{D}^*(0, T)$, just as in Proposition 2.1, and with the same proof of equivalence. This will be the case in our first result.

THEOREM 4.1. *Let the spaces $V, H, \mathcal{V}$ be as above and let V be separable. Assume the operator $\mathcal{A} : V \rightarrow V'$ is given such that its realization in $\mathcal{V}$ is type M , bounded and coercive with*

$$\mathcal{A}v(v) \geq \alpha \|v\|_{\mathcal{V}}^p , \quad v \in \mathcal{V} .$$

Then for each $f \in \mathcal{V}'$ and $u_0 \in H$ there is a solution of the Cauchy Problem (4.1).

PROOF. We shall use the Galerkin method to obtain a solution. Let $\{w_1, w_2, \dots\}$ be a basis for V and let V_m denote the linear span of $\{w_1, w_2, \dots, w_m\}$ for $m \geq 1$. Since $\mathcal{A}$ is demicontinuous $\mathcal{V} \rightarrow \mathcal{V}'$ (see Lemma II.2.2), it is demicontinuous $V \rightarrow V'$, and thus the restriction $V_m \rightarrow V'_m$ is continuous. For each $m \geq 1$ we obtain a solution of the system of ordinary differential equations

$$(4.2) \quad u_m(t) \in V_m : (u'_m(t), w_j) + \mathcal{A}(u_m(t))(w_j) = f(t)(w_j) , \quad 1 \leq j \leq m ,$$

with $u_m(0) = u_0^m \in V_m$, where the sequence $\{u_0^m\}$ in V converges in H to u_0 . Each u_m is extended to a maximal interval $[0, T_m]$ with $0 < T_m \leq T$ by the Cauchy-Peano Theorem in the finite-dimensional space V_m , and it satisfies the estimate

$$1/2|u_m(t)|_H^2 + \alpha \int_0^t \|u_m(s)\|^p ds \leq 1/2|u_0^m|_H^2 + \int_0^t \|f(s)\| \|u_m(s)\| ds, \quad 0 \leq t \leq T_m.$$

This shows each $T_m = T$ and that $\{u_m\}$ is bounded in $\mathcal{V}$ and in $L^\infty(0, T; H)$. There is a subsequence (denoted by $\{u_m\}$ again) for which $u_m \rightharpoonup u$ in $\mathcal{V}$, $\mathcal{A}(u_m) \rightharpoonup \xi$ in $\mathcal{V}'$ and $u_m(T) \rightharpoonup u^*$ in H . For each $j \geq 1$ and each $\varphi \in C^1[0, T]$ we obtain from (4.2)

$$\begin{aligned} & - \int_0^T (u_m(t), \varphi'(t)w_j) dt + \int_0^T (\mathcal{A}(u_m) - f)(\varphi(t)w_j) dt \\ & = (u_0^m, w_j)\varphi(0) - (u_m(T), w_j)\varphi(T), \end{aligned}$$

and letting $m \rightarrow \infty$ gives

$$- \int_0^T (u(t), \varphi'(t)w_j) dt + \int_0^T (\xi - f)(\varphi(t)w_j) dt = (u_0, w_j)\varphi(0) - (u^*, w_j)\varphi(T).$$

Since $\{w_j\}$ is dense in V it follows that

$$(4.3) \quad u \in \mathcal{V} : u' + \xi = f \text{ in } \mathcal{V}', \quad u(0) = u_0 \text{ in } H,$$

and $u(T) = u^*$; here we have used Proposition 1.1. The crucial point of the proof remains: to show $\xi = \mathcal{A}(u)$.

From (4.2) we obtain

$$\int_0^T \mathcal{A}(u_m)(u_m) dt = f(u_m) + 1/2|u_0^m|_H^2 - 1/2|u_m(T)|_H^2, \quad m \geq 1,$$

and by weak lower-semi-continuity of the norm follows

$$\limsup_{m \rightarrow \infty} \mathcal{A}(u_m)(u_m) \leq f(u) + 1/2|u_0|_H^2 - 1/2|u(T)|_H^2.$$

From (4.3) we obtain by Proposition 1.2

$$1/2|u(T)|_H^2 - 1/2|u_0|_H^2 + \xi(u) = f(u),$$

so we have demonstrated in the above that

$$u_m \rightharpoonup u \text{ in } \mathcal{V}, \quad \mathcal{A}(u_m) \rightharpoonup \xi \text{ in } \mathcal{V}' \text{ and } \limsup \mathcal{A}(u_m)(u_m) \leq \xi(u).$$

Since $\mathcal{A}$ is type M this shows $\mathcal{A}(u) = \xi$. □

If we had a uniqueness result for solutions it would follow that the original sequence $\{u_m\}$ converges weakly to u in $\mathcal{V}$. This is the case if $\mathcal{A}$ is monotone, for then it follows directly that any two solutions u_1, u_2 of the evolution equation satisfy $d/dt|u_1(t) - u_2(t)|^2 \leq 0$, $0 \leq t \leq T$, so with the same initial value $u_1(0) = u_2(0)$ they are necessarily equal on $[0, T]$. Furthermore, when $\mathcal{A}$ is monotone we obtain in the proof of Theorem 4.1 that $\lim_{m \rightarrow \infty} \langle \mathcal{A}(u_m) - \mathcal{A}(u), u_m - u \rangle = 0$, so it follows that $u_m \rightarrow u$ in $L^\infty(0, T; H)$ and that if $\mathcal{A}$ is strongly monotone then we obtain strong convergence $u_m \rightarrow u$ in $\mathcal{V}$.

Let's consider hypotheses directly on an operator $\mathcal{A} : V \rightarrow V'$ which lead to corresponding hypotheses on its realization in $\mathcal{V}$. We shall also use $\mathcal{A}$ to denote the realization in $\mathcal{V}$.

LEMMA 4.1. *If $\mathcal{A} : V \rightarrow V'$ is demicontinuous, then for each measurable $w : [0, T] \rightarrow V$, $\mathcal{A}(w(\cdot)) : [0, T] \rightarrow V'$ is measurable. If also $\mathcal{A}$ is bounded with*

$$\|\mathcal{A}v\| \leq C\|v\|^{p-1}, \quad v \in V,$$

then its realization $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}'$ is demicontinuous.

PROOF. If w_n is a measurable step function and $w_n \rightarrow w$, a.e. $t \in [0, T]$, then $\mathcal{A}(w_n)$ is a measurable step function converging a.e. weakly to $\mathcal{A}(w)$ in V' , so $\mathcal{A}(w)$ is (weakly = strongly) measurable in the separable V' . The verification of demicontinuity is likewise standard, and we leave it as an exercise. $\square$

LEMMA 4.2. *Let $\mathcal{A} : V \rightarrow V'$ and assume $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}'$. Then $\mathcal{A}$ is V -monotone if and only if its $\mathcal{V}$ realization is monotone. If $\mathcal{A}$ is bounded as in Lemma 4.1, then $\mathcal{A}$ is V -hemicontinuous if and only if its $\mathcal{V}$ realization is hemicontinuous.*

PROOF. If $\mathcal{A}$ is V -monotone and $u, v \in \mathcal{V}$ then $\int_0^T (\mathcal{A}(u(t)) - \mathcal{A}(v(t)))(u(t) - v(t)) dt \geq 0$ since the integrand is pointwise non-negative. Conversely, if we set $u(t) = u$, $v(t) = v \in V$ in this inequality we see $\mathcal{A}$ is V -monotone. To check hemicontinuity in $\mathcal{V}$, first let $u, v \in V$ and ε be given in $\mathbb{R}$. Then we obtain

$$\begin{aligned} |\mathcal{A}(u + \varepsilon v)(v)| &\leq C\|u + \varepsilon v\|^{p-1}\|v\| \\ &\leq C(\|u\| + \varepsilon\|v\|)^{p-1}\|v\| \\ &\leq C\left(\frac{(\|u\| + \varepsilon\|v\|)^p}{p'} + \frac{\|v\|^p}{p}\right). \end{aligned}$$

Thus, if $u, v \in \mathcal{V}$ and $|\varepsilon| \leq 1$ it follows

$$\left| \mathcal{A}(u(t) + \varepsilon v(t))v(t) \right| \leq C(\|u(t)\| + \|v(t)\|)^p + \|v(t)\|^p, \quad 0 \leq t \leq T,$$

and the right side is integrable, so there follows by the Lebesgue Theorem 1.4

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \mathcal{A}(u(t) + \varepsilon v(t))v(t) dt = \int_0^T \mathcal{A}(u(t))v(t) dt,$$

because $\mathcal{A}$ being V -hemicontinuous gives pointwise convergence of the integrand. The converse follows by letting u, v be constants, even without the condition of boundedness. $\square$

An immediate consequence is that Theorem 4.1 holds and the solution of (4.1) is unique if $\mathcal{A} : V \rightarrow V'$ is monotone, hemicontinuous, and bounded as in Lemma 4.1 and coercive with an estimate on V similar to that in Theorem 4.1. This can be extended to cover a family of such operators and the corresponding time-dependent Cauchy Problem. We describe this situation next and also relax the coercive hypothesis.

PROPOSITION 4.1. *Let the spaces $V, H, \mathcal{V}$ be given as before with V separable. Assume a family of operators $\mathcal{A}(t, \cdot) : V \rightarrow V'$, $0 \leq t \leq T$, is given such that*

- (i) *for each $v \in V$ the function $\mathcal{A}(\cdot, v) : [0, T] \rightarrow V'$ is measurable,*
- (ii) *for a.e. $t \in [0, T]$ the operator $\mathcal{A}(t, \cdot) : V \rightarrow V'$ is monotone, hemicontinuous and bounded by*

$$\|\mathcal{A}(t, v)\| \leq C(\|v\|^{p-1} + k(t)), \quad v \in V,$$

- where $k \in L^{p'}[0, T]$,
 (iii) there is a seminorm $[\cdot]$ on V and numbers $\lambda > 0$, $\alpha > 0$ such that

$$[v] + \lambda|v|_H \geq \alpha\|v\|, \quad \text{and} \\ \mathcal{A}(t, v)(v) \geq \alpha[v]^p, \quad \text{a.e. } t \in [0, T], \quad v \in V.$$

Then for each $f \in \mathcal{V}'$ and $u_0 \in H$ there exists a unique

$$(4.4) \quad u \in \mathcal{V} : u'(t) + \mathcal{A}(t, u(t)) = f(t) \text{ in } \mathcal{V}', \quad u(0) = u_0 \text{ in } H.$$

PROOF. With the same proofs as Lemma 4.1 and Lemma 4.2 we find $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}'$ is type M and bounded. Thus we obtain local solutions u_m of (4.2) as before. The proof of Theorem 4.1 will give the result claimed here if we can verify that $\{u_m\}$ is bounded as before by using (iii), and this we proceed to do. For simplicity we suppress the subscript m and let u denote the solution of (4.2). From the two estimates in (iii), Hölder's and then Young's inequalities, respectively, we obtain

$$\begin{aligned} & \frac{1}{2}|u(t)|_H^2 + \alpha \int_0^t [u(s)]^p ds \\ & \leq \frac{1}{2}|u_0|_H^2 + \int_0^t \|f(s)\| \|u(s)\| ds \\ & \leq \frac{1}{2}|u_0|_H^2 + 1/\alpha \int_0^t \|f(s)\| ([u(s)] + \lambda|u(s)|_H) ds \\ & \leq \frac{1}{2}|u_0|_H^2 + (1/\alpha)\|f\|_{\mathcal{V}'} \left(\left(\int_0^t [u]^p ds \right)^{1/p} + \lambda \left(\int_0^t |u|_H^p ds \right)^{1/p} \right) \\ & \leq \frac{1}{2}C(u_0, f, \alpha, \lambda, p) + \frac{\alpha}{2} \int_0^t [u]^p ds + \frac{1}{2} \left(\int_0^t |u|_H^p ds \right)^{2/p}. \end{aligned}$$

This gives us

$$(4.5) \quad |u(t)|_H^2 + \alpha \int_0^t [u(s)]^p ds \leq C(u_0, f, \alpha, \lambda, p) + \left(\int_0^t |u|_H^p ds \right)^{2/p}$$

and from this follows $|u(t)|_H^p \leq c(1 + \int_0^t |u|_H^p ds)$, so we obtain an explicit bound on $\|u\|_{L^\infty(0, T; H)}$ and then from (4.5) a bound on $\|u\|_{\mathcal{V}}$. $\square$

Finally we note that the coercive assumption can be relaxed in a more elementary way when the operators satisfy linear growth rates.

COROLLARY 4.1. Assume the situation of Proposition 4.1 with $p = 2$, but replace (iii) by the following:

- (iii') there is a pair $\lambda, \alpha > 0$ such that

$$\mathcal{A}(t, v)(v) + \lambda|v|_H^2 \geq \alpha\|v\|^2, \quad \text{a.e. } t \in [0, T], \quad v \in V.$$

Then (4.4) has a unique solution for each $f \in \mathcal{V}'$ and $u_0 \in H$.

PROOF. Note that an exponential shift, i.e., the change-of-variable $u(t) = e^{\lambda t} w(t)$, gives an equivalent problem for w in which $\mathcal{A}(t, u)$ is replaced by $\mathcal{A}(t, e^{\lambda t} u) + \lambda(u, \cdot)$ and $f(t)$ is replaced by $e^{-\lambda t} f(t)$. The new problem satisfies the hypothesis with $\lambda = 0$. $\square$

The next result shows that we obtain a *strong solution* when $\mathcal{A}$ is a derivative. This is a nonlinear analogue of Proposition 2.5 and it likewise corresponds to the solutions obtained by the semigroup on the pivot space H .

PROPOSITION 4.2. *Let the spaces $V, H, \mathcal{V}$ be given as before with V separable. Let the function $\varphi : V \rightarrow \mathbb{R}^+$ be convex and G -differentiable, $\varphi(0) = 0$, and set $\mathcal{A} \equiv \varphi' : V \rightarrow V'$. Assume that*

(i) *$\mathcal{A}$ is bounded by*

$$\|\mathcal{A}(v)\| \leq C(\|v\|^{p-1} + 1), \quad v \in V$$

(ii) *there is a seminorm $[\cdot]$ on V and numbers $\lambda > 0, \alpha > 0$ such that*

$$[v] + \lambda|v|_H \geq \alpha\|v\|, \quad \text{and}$$

$$\mathcal{A}(v)v \geq \alpha[v]^p, \quad v \in V.$$

Let $f \in \mathcal{V}'$, $u_0 \in H$, and u be the solution of (4.1).

(a) *If $t^{1/2}f \in L^2(0, T; H)$ then $t^{1/2}u' \in L^2(0, T; H)$ and thus*

$$u'(t) + \mathcal{A}(u(t)) = f(t) \quad \text{in } H, \quad \text{a.e. } t \in [0, T].$$

(b) *If $f \in L^2(0, T; H)$ and $u_0 \in V$ then $u' \in L^2(0, T; H)$.*

(c) *If also we assume*

$$\varphi(v) \geq \alpha[v]^p, \quad v \in V,$$

then $u \in L^\infty(0, T; V)$.

PROOF. From Proposition II.7.4 it follows that $\mathcal{A}$ is monotone and hemicon-
tinuous, so the existence and uniqueness of u are obtained, and u is the limit of the
sequence of solutions $\{u_m\}$ of the approximating problems (4.2). The additional
properties of u will be obtained from corresponding additional a-priori estimates on
 $\{u_m\}$. If $j_m : V_m \rightarrow V$ is the usual injection and its dual, the restriction operator,
is denoted by $j'_m : V' \rightarrow V'_m$, it follows directly that $\varphi \circ j_m$ is G -differentiable and
 $(\varphi \circ j_m)' = j'_m \circ \varphi' \circ j_m$. (See Proposition II.7.8 also.) Thus in the finite dimensional
space V_m we have

$$\frac{d}{dt} \varphi(u_m(t)) = \left(\mathcal{A}(u_m(t)), u'_m(t) \right)_H.$$

(a) Take the H -scalar-product of (4.2) with $tu'_m(t)$ and integrate; this leads to

$$\int_0^T t |u'_m(t)|_H^2 dt + 2T\varphi(u_m(T)) \leq \int_0^T t |f(t)|_H^2 dt + 2 \int_0^T \varphi(u_m(t)) dt.$$

The second term on the left is non-negative and the second term on the right is bounded by

$$2 \int_0^T \mathcal{A}(u_m(t)) u_m(t) dt \leq 2 \|\mathcal{A}(u_m)\|_{\mathcal{V}'} \|u_m\|_{\mathcal{V}},$$

- so it follows that $\{t^{1/2}u'_m\}$ converges weakly in $L^2(0, T; H)$ to $t^{1/2}u'$.
- (b) Take the H -scalar-product of (4.2) with $u'_m(t)$ and integrate to obtain likewise

$$\int_0^T |u'_m(t)|_H^2 dt + 2\varphi(u_m(T)) \leq 2\varphi(u_0^m) + \int_0^T |f(t)|_H^2 dt .$$

- Since $u_0 \in V$ we choose $u_0^m \rightarrow u_0$ in V and then we find $\{u'_m\}$ is bounded and, hence, converges weakly to u' in $L^2(0, T; H)$.
- (c) In the above we integrate over $[0, t]$, $0 < t \leq T$, and this shows $\{\varphi(u_m)\}$ is bounded in $L^\infty[0, T]$. $\square$

We shall present some examples of initial-boundary-value problems for nonlinear parabolic partial differential equations which can be resolved as direct applications of the preceding results. Details will be given for an instructive model problem from which obvious generalizations can be constructed easily as in Sections II.5 and II.8. Also we shall indicate certain useful extensions of the method obtained by minor modifications of the proofs or the choice of spaces in which the solution is sought.

For simplicity we begin with the following model problem. Let G be a bounded domain in $\mathbb{R}^n$ whose boundary ∂G is a C^1 manifold, $1 < p < \infty$, and $a(\xi) = |\xi|^{p-1} \operatorname{sgn} \xi$, $b(\cdot) \in L^\infty(G)$ and $c(\cdot) \in L^\infty(\partial G)$ be non-negative, and then define

$$\begin{aligned} \mathcal{A}u(v) = \int_G \left\{ \sum_{j=1}^n a(\partial_j u(x)) \partial_j v(x) + b(x) a(u(x)) v(x) \right\} dx \\ + \int_{\partial G} c(s) a(u(s)) v(s) ds , \quad u, v \in W^{1,p}(G) . \end{aligned}$$

The second integral depends on $u(s) = \gamma u(s)$, the trace $\gamma u \in L^p(\partial G)$ of u and likewise of $v \in W^{1,p}(G)$. Let $\Omega = G \times (0, T)$, $\Gamma = \partial G \times (0, T)$, $F \in L^{p'}(\Omega)$, $F_0 \in L^{p'}(\Gamma)$, and set

$$f(t)v = \int_G F(x, t)v(x) dx + \int_{\partial G} F_0(s, t)v(s) ds , \quad 0 < t < T , \quad v \in W^{1,p}(G) .$$

In order to obtain various types of boundary conditions we shall consider restrictions of $\mathcal{A}$, $f(t)$ to corresponding closed subspaces V with $W_0^{1,p}(G) \subset V \subset W^{1,p}(G)$. Also we take $H = L^2(G)$, so $V \subset H$ if we require $p \geq 2$. (By the Sobolev imbedding Theorem II.4.3 it is sufficient for $p > 2n/(n+2)$, but even this will be relaxed below.)

EXAMPLE 4.A. DIRICHLET BOUNDARY CONDITIONS. We choose $V = W_0^{1,p}(G)$, and $c \equiv F_0 \equiv 0$. If $u_0 \in L^2(G)$, it follows from Proposition 4.1 and II.5 that there exists a unique generalized solution of

$$(4.6.a) \quad \partial_t u - \sum_{j=1}^n \partial_j a(\partial_j u) + ba(u) = F , \quad (x, t) \in \Omega ,$$

$$(4.6.b) \quad u(s, t) = 0 , \quad (s, t) \in \Gamma ,$$

$$(4.6.c) \quad u(x, 0) = u_0(x) , \quad x \in G .$$

The *quasilinear parabolic* equation (4.6.a) holds with each term in $\mathcal{V}' = L^{p'}(0, T; V')$, $V' = W^{-1,p'}(G)$, the boundary condition (4.6.b) follows in the sense of trace since $u(\cdot, t) \in W_0^{1,p}(G)$, and the initial condition (4.6.c) is meaningful because $\lim_{t \rightarrow 0} u(\cdot, t) = u_0$ in $L^2(G)$.

A corresponding non-homogeneous boundary condition can be obtained as follows. Let

$$g \in L^p(0, T; W^{1,p}(G)) \quad , \quad g' \in L^{p'}(0, T; W^{-1,p'}(G))$$

be any such function which attains the desired boundary conditions. Define $\mathcal{A}(t, w) = \mathcal{A}(w + g(t))$ for $w \in V$ and $0 \leq t \leq T$; note that

$$\|\mathcal{A}(t, w)\|_{V'} \leq C 2^{p-1} \left(\|w\|_V^{p-1} + \|g(t)\|_V^{p-1} \right) ,$$

since $\mathcal{A}$ is bounded as in Lemma 4.1, so $\mathcal{A}(t, \cdot)$ fulfills the hypotheses of Proposition 4.1. Let w be the unique solution of (4.4), i.e.,

$$\begin{aligned} w \in L^p(0, T; W_0^{1,p}) : w' + \mathcal{A}(w + g) &= f - g' \text{ in } L^{p'}(0, T; W^{-1,p'}) , \\ w(0) &= u_0 - g(0) \text{ in } L^2(G) . \end{aligned}$$

Then $u(t) \equiv w(t) + g(t)$ satisfies

$$\begin{aligned} u \in L^p(0, T; W^{1,p}) : u' + \mathcal{A}(u) &= f \text{ in } L^{p'}(0, T; W^{-1,p'}) , \\ u(t) - g(t) &\in W_0^{1,p} , \quad 0 < t \leq T , \quad u(0) = u_0 \text{ in } L^2(G) . \end{aligned}$$

That is, u is the generalized solution of (4.6a), (4.6c) and $\gamma u(t) = \gamma g(t)$, $0 < t \leq T$, the problem with prescribed boundary values $\gamma g(t)$. Note finally that $\mathcal{A}$ as given above is the G -differential of the continuous convex

$$\Phi(u) = \int_G \left\{ \sum_{j=1}^n \varphi(\partial_j u(x)) + b(x) \varphi(u(x)) \right\} dx + \int_{\partial G} c(s) \varphi(u(s)) ds$$

where $\varphi(\xi) = |\xi|^p/p$; see Proposition II.7.6 and Example II.8.D. Thus the regularity results of Proposition 4.2 apply to (4.6).

EXAMPLE 4.B. COMPLEMENTARY BOUNDARY CONDITIONS. We choose $V = W^{1,p}(G)$ and let $\mathcal{A}$, f be given as above. If $u_0 \in L^2(G)$ it follows from Proposition 4.1 and the formal computations of II.5 that there exists a unique generalized solution of

$$(4.7.a) \quad \partial_t u - \sum_{j=1}^n \partial_j a(\partial_j u) + b a(u) = F , \quad (x, t) \in \Omega ,$$

$$(4.7.b) \quad \sum_{j=1}^n a(\partial_j u) \nu_j + c a(u) = F_0 , \quad (s, t) \in \Gamma ,$$

$$(4.7.c) \quad u(x, 0) = u_0(x) , \quad x \in G .$$

The boundary condition (4.7.b) is of *Neumann* or *Robin* type and is only formally satisfied. However, Proposition 4.2 puts each term of (4.7.a) in $L^2(G) \equiv H$ at a.e. $t \in [0, T]$ when $F_0 \equiv 0$ and $t^{1/2} F(\cdot, t) \in L^2(\Omega)$. Then the abstract Green's formula (II.5.7) applies directly to this situation and gives a precise meaning to (4.7.b).

Alternatively, this can be achieved from the regularity theory of nonlinear elliptic equations in divergence form when the region G is sufficiently smooth.

This example can be extended in various ways. For instance, if $1 < p < 2$, the inclusion $W^{1,p}(G) \subset L^2(G)$ is not necessarily true. But if we define $V = W^{1,p}(G) \cap L^2(G)$ and $\|v\| = [v] + |v|$ with $[v] = \|v\|_{W^{1,p}}$, $|v| = \|v\|_{L^2}$, then we have $V \hookrightarrow H = L^2(G)$ and Propositions 4.1 and 4.2 are applicable.

III.5. Abstract Difference Approximations

The underlying idea in this section is to approximate the differential equation

$$\frac{du(t)}{dt} + \mathcal{A}(u(t)) = f(t) , \quad 0 \leq t < T ,$$

by a *finite-difference* equation

$$\frac{1}{\varepsilon}(u(t) - u(t - \varepsilon)) + \mathcal{A}(u(t)) = f(t) , \quad 0 < \varepsilon \leq t < T .$$

In order to implement this approximation, we introduce the *semigroup of translations* $S(\varepsilon)u(t) \cong u(t - \varepsilon)$ by which we can write our finite-difference equation in the form

$$\frac{1}{\varepsilon}(I - S(\varepsilon))u(t) + \mathcal{A}(u(t)) = f(t) .$$

For the Cauchy Problem this reduces to a family of stationary problems, since $S(\varepsilon)u(t)$ involves “preceding times” and so in principle is known. To solve the finite-difference equation we need to be able to invert the operator $I + \varepsilon\mathcal{A}$ for all $\varepsilon > 0$. This is certainly a natural requirement for monotone operators $\mathcal{A}$, and we shall find the preceding approximation is a very useful technique.

Let $\mathcal{V}$ be a separable reflexive Banach space which is dense and continuous in a Hilbert space $\mathcal{H}$. Identify $\mathcal{H}$ with its dual so we obtain $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$. Suppose that $\{S(t) : t \geq 0\}$ is a contraction semigroup on $\mathcal{V}'$ and on $\mathcal{H}$. Denote its generator on $\mathcal{V}'$ and $\mathcal{H}$ by $-L$ and $-L_{\mathcal{H}}$, respectively. These are given on the domains

$$D(L) = \{g \in \mathcal{V}' : \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon}(I - S(\varepsilon))g = Lg \in \mathcal{V}'\}$$

$$D(L_{\mathcal{H}}) = \{g \in \mathcal{H} : \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon}(I - S(\varepsilon))g = L_{\mathcal{H}}g \in \mathcal{H}\}$$

with values as indicated and the limits taken in the norms of the corresponding spaces, and we have

$$(L_{\mathcal{H}}g, g)_{\mathcal{H}} \geq 0 , \quad g \in D(L_{\mathcal{H}}) ,$$

since each $S(\varepsilon)$ is a contraction in $\mathcal{H}$. We denote the semigroup of adjoint or dual operators by $\{S^*(t) : t \geq 0\}$ in $\mathcal{V}$ and $\mathcal{H}$, respectively; their corresponding generators are $-L^*$ and $-L_{\mathcal{H}}^*$, the respective adjoint or dual operators of $-L$ and $-L_{\mathcal{H}}$ as unbounded operators. Finally, we shall regard L as an operator from $\mathcal{V}$ to $\mathcal{V}'$ with domain $\mathcal{V} \cap D(L)$. As such it is monotone, since

$$\left\langle \frac{1}{\varepsilon}(I - S(\varepsilon))v, v \right\rangle = \left(\frac{1}{\varepsilon}(I - S(\varepsilon))v, v \right)_{\mathcal{H}} \geq 0 , \quad v \in \mathcal{V} ,$$

because $S(\varepsilon)$ is a contraction on $\mathcal{H}$, and letting $\varepsilon \rightarrow 0$ shows

$$Lv(v) \geq 0 , \quad v \in \mathcal{V} \cap D(L) .$$

REMARK. If in addition $\{S(t) : t \geq 0\}$ is a contraction semigroup on $\mathcal{V}$, then $\mathcal{V} \cap D(L)$ is dense in $\mathcal{V}$. This follows since for each $v \in \mathcal{V}$, we have

$$(I + \varepsilon L)^{-1}v \rightarrow v \text{ in } \mathcal{V} \text{ as } \varepsilon \downarrow 0.$$

THEOREM 5.1 (BARDOS-BREZIS). *Let the spaces $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ and semigroup generator $-L$ be given as above. Let $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}'$ be type M , bounded and coercive. Then for each $f \in \mathcal{V}'$ there exists a*

$$(5.1) \quad u \in \mathcal{V} \cap D(L) : Lu + \mathcal{A}(u) = f \text{ in } \mathcal{V}'.$$

PROOF. Let $\varepsilon > 0$. Since $I - S(\varepsilon)$ is continuous, linear and monotone on $\mathcal{H}$, it is likewise from $\mathcal{V}$ to $\mathcal{V}'$ and, hence, $\frac{1}{\varepsilon}(I - S(\varepsilon)) + \mathcal{A}$ is type M , bounded and coercive; see Example II.2.C. Thus, for each $\varepsilon > 0$ there exists a solution of the “stationary” problem

$$(5.2) \quad u_\varepsilon \in \mathcal{V} : \frac{1}{\varepsilon}(I - S(\varepsilon))u_\varepsilon + \mathcal{A}(u_\varepsilon) = f \text{ in } \mathcal{V}'$$

by Theorem II.2.1. Taking the value of (5.2) at u_ε shows that

$$\mathcal{A}u_\varepsilon(u_\varepsilon) \leq f(u_\varepsilon), \quad \varepsilon > 0.$$

Since $\mathcal{A}$ is coercive and bounded, it follows that $\{u_\varepsilon\}$ is bounded in $\mathcal{V}$ and $\{\mathcal{A}(u_\varepsilon)\}$ is bounded in $\mathcal{V}'$. By passing to a subsequence, which we hereafter denote by $\{u_\varepsilon\}$, we have $u_\varepsilon \rightharpoonup u$ in $\mathcal{V}$ and $\mathcal{A}(u_\varepsilon) \rightharpoonup g$ in $\mathcal{V}'$.

Next we show these satisfy an equality like (5.1). For each $v \in \mathcal{V}$ we find

$$\langle f - \mathcal{A}(u_\varepsilon), v \rangle = \frac{1}{\varepsilon}(I - S(\varepsilon))u_\varepsilon(v) = \langle u_\varepsilon, \frac{1}{\varepsilon}(I - S^*(\varepsilon))v \rangle,$$

and, by letting $\varepsilon \rightarrow 0$, we get

$$\langle f - g, v \rangle = \langle u, L^*v \rangle, \quad v \in D(L^*).$$

From here we obtain for each $v \in \mathcal{V}$

$$\langle f - g, (I + \varepsilon L^*)^{-1}v \rangle = \langle L(I + \varepsilon L)^{-1}u, v \rangle.$$

Since $(I + \varepsilon L^*)^{-1}v$ is bounded in $\mathcal{V}$, it follows from the Uniform Boundedness Theorem II.1.2 that $\{L(I + \varepsilon L)^{-1}u : \varepsilon > 0\}$ is bounded in $\mathcal{V}'$. This implies $u \in D(L)$ and

$$\langle f - g, v \rangle = \langle Lu, v \rangle, \quad v \in \mathcal{V},$$

(see Section I.5 or Theorem IV.1.1) so we obtain

$$u \in D(L) : Lu + g = f \text{ in } \mathcal{V}'.$$

Note that the preceding also shows $\frac{1}{\varepsilon}(I - S(\varepsilon))u_\varepsilon \rightharpoonup Lu$ in $\mathcal{V}'$.

It remains only to prove $\mathcal{A}(u) = g$. But we have $u_\varepsilon \rightharpoonup u$, $\mathcal{A}(u_\varepsilon) \rightharpoonup g$ and

$$\begin{aligned} \limsup \mathcal{A}(u_\varepsilon)(u_\varepsilon - u) &= \lim f(u_\varepsilon - u) - \liminf \langle \frac{1}{\varepsilon}(I - S(\varepsilon))u_\varepsilon, u_\varepsilon - u \rangle \\ &\geq -\lim \langle \frac{1}{\varepsilon}(I - S(\varepsilon))u, u_\varepsilon - u \rangle = 0, \end{aligned}$$

and the result follows since $\mathcal{A}$ is type M . □

Theorem 5.1 gives a *weak solution* of (5.1), a solution for which each term of (5.1) belongs to $\mathcal{V}'$. Under an additional hypothesis we obtain a *strong solution*, one for which each term is in $\mathcal{H}$.

COROLLARY 5.1. *Assume further that $\mathcal{V}$ is invariant under the contraction semigroup $\{S(t) : t \geq 0\}$ and that*

$$\langle \mathcal{A}v, S(t)v \rangle \leq \mathcal{A}v(v) , \quad v \in \mathcal{V} , \quad t > 0 .$$

Then for each $f \in \mathcal{H}$ there exists a

$$(5.3) \quad u \in \mathcal{V} \cap D(L_{\mathcal{H}}) : L_{\mathcal{H}}u + \mathcal{A}(u) = f \quad \text{in } \mathcal{H} .$$

PROOF. Let u_{ε} be a solution of (5.2); evaluate this equation on $\frac{1}{\varepsilon}(I - S(\varepsilon))u_{\varepsilon} \in \mathcal{V}$ to obtain

$$\left\| \frac{1}{\varepsilon}(I - S(\varepsilon))u_{\varepsilon} \right\|_{\mathcal{H}}^2 + \frac{1}{\varepsilon} \langle \mathcal{A}(u_{\varepsilon}), (I - S(\varepsilon))u_{\varepsilon} \rangle = \langle f, \frac{1}{\varepsilon}(I - S(\varepsilon))u_{\varepsilon} \rangle .$$

The second term is non-negative, so it follows that $\{\frac{1}{\varepsilon}(I - S(\varepsilon))u_{\varepsilon}\}$ and $\{\mathcal{A}(u_{\varepsilon})\}$ are both bounded in $\mathcal{H}$. $\square$

REMARKS. If $\mathcal{A}$ is strictly-monotone there is exactly one solution of (5.1) and Corollary 5.1 is a *regularity* result. Note also the similarity to Proposition 4.2 where the additional estimates were obtained by “multiplying” the equation by the derivative. Here we “multiplied” by the finite-difference to obtain the extra estimate.

Next we shall apply Theorem 5.1 to various problems for evolution equations.

PROPOSITION 5.1. *Let V be separable reflexive Banach space, dense and continuous in a Hilbert space H which is identified with its dual, so $V \subset H \subset V'$. Let $p \geq 2$ and set $\mathcal{V} = L^p(0, T; V)$, $\mathcal{H} = L^2(0, T; H)$. Assume a family of operators $\mathcal{A}(t, \cdot) : V \rightarrow V'$, $0 \leq t \leq T$, is given such that*

- (i) *for each $v \in V$ the function $\mathcal{A}(\cdot, v) : [0, T] \rightarrow V'$ is measurable,*
- (ii) *for a.e. $t \in [0, T]$ the operator $\mathcal{A}(t, \cdot) : V \rightarrow V'$ is monotone, hemicontinuous and bounded by*

$$\|\mathcal{A}(t, v)\| \leq C(\|v\|^{p-1} + k(t)) , \quad v \in V , \quad 0 \leq t \leq T ,$$

where $k \in L^{p'}[0, T]$, and

- (iii) *there is an $\alpha > 0$ such that*

$$\mathcal{A}(t, v)(v) \geq \alpha\|v\|^p - k(t) , \quad v \in V , \quad 0 \leq t \leq T .$$

Then for each $f \in \mathcal{V}'$ and $u_0 \in H$ there exists a unique solution of the Cauchy problem

$$(5.4) \quad u \in \mathcal{V} : u'(t) + \mathcal{A}(t, u(t)) = f(t) \quad \text{in } \mathcal{V}' , \quad u(0) = u_0 .$$

PROOF. Define $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}'$ as the realization of $\mathcal{A}(t, \cdot)$, i.e., $\mathcal{A}(v)(t) = \mathcal{A}(t, v(t))$, a.e. $t \in [0, T]$. From the discussion of Section 4 it follows that $\mathcal{A}$ is type M , bounded and coercive. Define the *semigroup of translations*

$$S(s)v(t) = \begin{cases} 0, & 0 \leq t \leq s, \\ v(t-s), & s < t \leq T, \end{cases} \quad s \geq 0,$$

which operates in each of $\mathcal{V}$, $\mathcal{H}$ and $\mathcal{V}'$. The generator $-L$ in $\mathcal{V}'$ is determined by

$$L = \frac{d}{dt}, \quad D(L) = \{g \in W^{1,p'}(0, T; V') : g(0) = 0\}.$$

Note that $\mathcal{V} \cap D(L) \subset C(0, T; H)$ by Proposition 1.2. With this choice of data, Theorem 5.1 yields the desired result for the special case $u_0 = 0$. If u_0 is given as above, let w_0 be any function which satisfies $w_0 \in \mathcal{V}$, $w'_0 \in \mathcal{V}'$ and $w_0(0) = u_0$. Then u is a solution of (5.4) if and only if $w \equiv u - w_0$ satisfies

$$v \in \mathcal{V} : w' + \mathcal{A}(w + w_0) = f - w'_0 \text{ in } \mathcal{V}', \quad w(0) = 0,$$

and this special case is resolved as above. $\square$

COROLLARY 5.2. *Let the spaces $V \subset H \subset V'$ and $\mathcal{V}$, $\mathcal{H}$ be given as above. Let the function $\varphi : V \rightarrow \mathbb{R}^+$ be convex and G -differentiable, $\varphi(0) = 0$, and set $\mathcal{A} \equiv \varphi' : V \rightarrow V'$. Assume that*

(i) $\mathcal{A}$ is bounded by

$$\|\mathcal{A}(v)\| \leq C(\|v\|^{p-1} + 1), \quad v \in V,$$

(ii) there is an $\alpha > 0$ such that

$$\mathcal{A}v(v) \geq \alpha\|v\|^p, \quad v \in V.$$

If $f \in \mathcal{H}$ and $u_0 \in V$, then the solution of the Cauchy Problem (5.4) satisfies $u' \in \mathcal{H}$ and $\mathcal{A}(u) \in \mathcal{H}$.

PROOF. $\mathcal{A} : V \rightarrow V'$ is monotone and hemicontinuous so we are in the situation of Proposition 5.1. It remains only to verify the extra hypothesis of Corollary 5.1. If $v \in \mathcal{V}$ and $v_\varepsilon = S(\varepsilon)v$, then at a.e. $t \in [0, T]$ we have

$$\mathcal{A}v(t)(v_\varepsilon(t) - v(t)) \leq \varphi(v_\varepsilon(t)) - \varphi(v(t))$$

since $\mathcal{A} = \varphi'$, and from this follows

$$\begin{aligned} \mathcal{A}v(v_\varepsilon) &= \int_\varepsilon^T \mathcal{A}(v(t))(v_\varepsilon(t)) dt \\ &\leq \mathcal{A}v(v) + \int_\varepsilon^T \varphi(v(t-\varepsilon)) dt - \int_0^T \varphi(v(t)) dt \\ &\leq \mathcal{A}v(v) \end{aligned}$$

as desired. $\square$

The preceding discussion shows that Theorem 5.1 and Corollary 5.1 yield as special cases Propositions 4.1 and 4.2, respectively, on the weak and strong solvability of the Cauchy Problem (5.4). Note that Corollary 5.2 extends easily to the case

$$\mathcal{A}(t)v \equiv \frac{d}{dv}\varphi(t, v), \quad 0 \leq t \leq T, \quad v \in V,$$

where $t \mapsto \varphi(t, v)$ is *non-increasing* for each $v \in V$. Other useful applications of Theorem 5.1 can be given. Here is an example.

PROPOSITION 5.2. *Let the spaces V , H , $\mathcal{V}$, $\mathcal{H}$ and operators $\mathcal{A}(t, \cdot)$ be given as in Proposition 5.1. Then for each $f \in \mathcal{V}'$ there exists a solution of the Periodic Problem*

$$(5.5) \quad u \in \mathcal{V} : u'(t) + \mathcal{A}(t, u(t)) = f(t) \text{ in } \mathcal{V}' , \quad u(0) = u(T) .$$

PROOF. Define the *periodic semigroup*

$$S(s)v(t) = \begin{cases} v(T - s + t) , & 0 < t < s , \\ v(t - s) , & s < t < T , \end{cases} \quad s \geq 0$$

which operates in $\mathcal{V}$, $\mathcal{H}$ and $\mathcal{V}'$. Its generator $-L$ is determined by

$$L = \frac{d}{dt} , \quad D(L) = \{g \in W^{1,p'}(0, T; V') : g(0) = g(T)\} ,$$

so (5.5) is obtained as (5.1). $\square$

COROLLARY 5.3. *If $\mathcal{A}$ is given as in Corollary 5.2 and $f \in \mathcal{H}$, there exists a solution u of (5.5) with $u \in W^{1,2}(0, T; H)$.*

PROOF. Again we need only verify the estimate

$$\begin{aligned} \langle \mathcal{A}v, S(s)v \rangle &\leq \int_0^T \varphi(S(s)v(t)) dt \\ &\quad - \int_0^T \varphi(v(t)) dt + \mathcal{A}v(v) , \quad v \in \mathcal{V} , s \geq 0 \end{aligned}$$

which follows since $\mathcal{A} = \varphi'$, and then note that the right side equals $\mathcal{A}v(v)$ because the semigroup is periodic. Thus Corollary 5.3 follows from Corollary 5.1. $\square$

Another application is the *history-value problem*.

PROPOSITION 5.3. *Let the spaces V , H and operators $\mathcal{A}(t, \cdot)$ be given as in Proposition 5.1 for all $t \leq 0$ and assume $p = 2$; set $\mathcal{V} = L^2(-\infty, 0; V)$ and $\mathcal{H} = L^2(-\infty, 0; H)$. Then for each $f \in \mathcal{V}'$ there exists a solution of the singular Cauchy problem*

$$(5.6) \quad u \in \mathcal{V} : u'(t) + \mathcal{A}(t, u(t)) = f(t) \text{ in } \mathcal{V}' = L^2(-\infty, 0; V') .$$

COROLLARY 5.4. *If $\mathcal{A}$ is given as in Corollary 5.2 and $f \in \mathcal{H}$, there exists a solution of (5.6) with $u \in W^{1,2}(-\infty, 0; H)$.*

As before these are consequences of Theorem 5.1 and Corollary 5.1; this follows from the next result which characterizes the appropriate semigroup and generator.

LEMMA. *Define $L = \frac{d}{dt}$ on $\mathcal{V}'$ with domain $D(L) = W^{1,2}(-\infty, 0; V')$. For each $\varepsilon > 0$, $(I + \varepsilon L)^{-1}$ is a contraction on $\mathcal{V}'$, and the contraction semigroup on $\mathcal{V}'$ generated by $-L$ is given by*

$$S(s)v(t) = v(t - s) , \quad -\infty < t \leq 0 \leq s .$$

For each $u \in D(L)$, $\lim_{t \rightarrow -\infty} u(t) = 0$, and $(I + \varepsilon L)u = f$ if and only if

$$(5.7) \quad u(t) = \frac{1}{\varepsilon} \int_{-\infty}^t f(s) e^{s-t/\varepsilon} ds, \quad -\infty < t < 0.$$

PROOF. Let $\varepsilon > 0$ and $(I + \varepsilon L)u = f$. Then

$$\begin{aligned} e^{t/\varepsilon} u(t) - e^{s/\varepsilon} u(s) &= \frac{1}{\varepsilon} \int_s^t f(\tau) e^{\tau/\varepsilon} d\tau, \quad s < t < 0, \\ \|e^{t/\varepsilon} u(t) - e^{s/\varepsilon} u(s)\|^2 &\leq \int_s^t \|f(\tau)\|^2 d\tau \frac{1}{2\varepsilon} (e^{2t/\varepsilon} - e^{2s/\varepsilon}) \end{aligned}$$

so the limit $\lim_{t \rightarrow -\infty} e^{t/\varepsilon} u(t) = c$ exists in V' and we have

$$u(t) = ce^{-t/\varepsilon} + \frac{1}{\varepsilon} \int_{-\infty}^t f(s) e^{(s-t)/\varepsilon} ds.$$

We shall show below that the integral belongs to $\mathcal{V}'$; the first term is in $\mathcal{V}'$ only if $c = 0$, so we obtain (5.7) for $u \in D(L)$.

Next let $f \in \mathcal{V}'$, $\varepsilon > 0$ and define u by (5.7). The integral converges because $f \in \mathcal{V}'$ and $e^{2s/\varepsilon}$ is integrable on $(-\infty, t)$. Note that we have

$$\frac{1}{\varepsilon} \int_{-\infty}^t e^{(s-t)/\varepsilon} ds = 1,$$

so the convexity of the squared-norm implies

$$(5.8) \quad \|u(t)\|^2 \leq \int_{-\infty}^t \|f(s)\|^2 \left\{ \frac{1}{\varepsilon} e^{(s-t)/\varepsilon} \right\} ds.$$

From Fubini's Theorem there follows

$$\|u\|_{\mathcal{V}'}^2 \leq \int_{-\infty}^0 \int_s^0 \|f(s)\|^2 \frac{1}{\varepsilon} e^{(s-t)/\varepsilon} dt ds \leq \|f\|_{\mathcal{V}'}^2,$$

so $u \in D(L)$ and $\|u\|_{\mathcal{V}'} \leq \|f\|_{\mathcal{V}'}$. Finally, from (5.8) follows

$$\varepsilon \|u(t)\|^2 \leq \int_{-\infty}^t \|f(s)\|^2 ds, \quad t < 0,$$

so $u(-\infty) = 0$.

To get the representation for the semigroup, note that for $v \in D(L)$ the function $v(t, s) = S(s)v(t)$ is the solution of

$$\begin{aligned} \frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} &= 0, & \text{a.e. } t \leq 0, \ s \geq 0, \\ v(t, 0) &= v(t), & \text{a.e. } t \leq 0, \end{aligned}$$

so we obtain $v(t, s) = v(t - s)$. □

We call (5.6) a *singular Cauchy problem* because the “initial condition” is implicit in the spaces which specify the solution, namely, $u(-\infty) = 0$. Only the zero

“initial value” can be attained. Moreover, Proposition 5.3 applies to the *singular evolution equation*

$$(5.9) \quad \frac{1}{b(s)} \frac{dw(s)}{ds} + \mathcal{A}(s, w(s)) = g(s) , \quad -\infty \leq s_0 < s < s_1 ,$$

where $b(\cdot)$ is a locally-integrable positive function on the interval (s_0, s_1) . The effect of the singularity at s_0 and at s_1 can be determined by the change-of-variable $t = B(s)$ where $B(\cdot)$ is an absolutely continuous primitive of $b(\cdot)$ on (s_0, s_1) . That is, w is a solution of (5.9) if and only if the function $u \equiv w \circ B^{-1}$ satisfies

$$\frac{du(t)}{dt} + \mathcal{A}(B^{-1}(t), u(t)) = g(B^{-1}(t)) , \quad B(s_0) < t < B(s_1) .$$

If $B(s_0) > -\infty$, then the Cauchy problem is appropriate: the value of $w(s_0)$ should be specified for the solution of (5.9). If $B(s_0) = -\infty$, then (5.9) is resolved by Proposition 5.3 with $w(s_0) = 0$.

Our final applications of Theorem 5.1 concern *parabolic systems* of equations.

PROPOSITION 5.4. *Let V_1 and V_2 be separable reflexive Banach spaces, each dense and continuous in a Hilbert space H identified with its dual. Let $p_1, p_2 \geq 2$ and set $\mathcal{V} = L^{p_1}(0, T; V_1) \times L^{p_2}(0, T; V_2)$, $\mathcal{H} = L^2(0, T; H) \times L^2(0, T; H)$. Let the functions $\varphi_j : V_j \rightarrow \mathbb{R}^+$ be convex and G -differentiable, $\varphi_j(0) = 0$, and define $\mathcal{A}_j = \varphi'_j : V_j \rightarrow V'_j$ for $j = 1, 2$. Assume each $\mathcal{A}_j$ is bounded and coercive on V_j with growth-order p_j (as in Corollary 5.2) for $j = 1, 2$. Let $a, b \in \mathbb{R}$ with $b \geq 0$. Then for each pair $[f_1, f_2] \in \mathcal{V}'$ and each pair $[u_0, u_T] \in H \times H$ there exists a solution of the system*

$$\begin{aligned} (5.10.a) \quad & u_1 \in L^{p_1}(0, T; V_1) : u'_1 + \mathcal{A}_1(u_1) + au_2 + b(u_1 - u_2) = f_1 \\ & \text{in } L^{p'_1}(0, T; V'_1) , \\ (5.10.b) \quad & u_2 \in L^{p_2}(0, T; V_2) : -u'_2 + \mathcal{A}_2(u_2) - au_1 + b(u_2 - u_1) = f_2 \\ & \text{in } L^{p'_2}(0, T; V'_2) , \\ (5.10.c) \quad & u_1(0) = u_0 , \quad u_2(T) = u_T \quad \text{in } H . \end{aligned}$$

Uniqueness holds if $\mathcal{A}_1, \mathcal{A}_2$ are strictly monotone.

COROLLARY 5.5. *Assume further that*

$$f_1, f_2 \in L^2(0, T; H) , \quad u_0 \in V_1 , \quad u_T \in V_2$$

and that $a = 0$. Then each term in (5.10.a) and (5.10.b) belongs to $L^2(0, T; H)$; that is,

$$u_1, u_2 \in W^{1,2}(0, T; H) .$$

PROOFS. The semigroup given on $\mathcal{V}'$ by $S(s)[g_1, g_2] = [g_1^s, g_2^s]$, where

$$g_1^s(t) = \begin{cases} 0 , & 0 < t < s \\ g_1(t-s) , & s < t < T \end{cases} , \quad g_2^s(t) = \begin{cases} g_2(t+s) , & 0 < t < T-s \\ 0 , & T-s < t < T \end{cases}$$

is generated by $-L$, where $L = [\frac{d}{dt}, -\frac{d}{dt}]$ on

$$D(L) = \{[g_1, g_2] \in W^{1,p'_1}(0, T; V'_1) \times W^{1,p'_2}(0, T; V'_2) : g_1(0) = g_2(T) = 0\} ,$$

so Theorem 5.1 gives the first result for $u_0 = u_T = 0$. The general case follows as in Proposition 5.1. Consider the function

$$\varphi([u_1, u_2]) = \int_0^T \left(\varphi_1(u_1(t)) + \varphi_2(u_2(t)) + (b/2)|u_1(t) - u_2(t)|_H^2 \right) dt, \quad [u_1, u_2] \in \mathcal{V}.$$

It is convex, G -differentiable and satisfies

$$\varphi(S(s)[u_1, u_2]) \leq \varphi([u_1, u_2]), \quad [u_1, u_2] \in \mathcal{V}, \quad s \geq 0,$$

so it follows as in the proof of Corollary 5.2 that Corollary 5.1 applies here.

REMARKS. Corollary 5.5 can also be obtained by applying Corollary 5.2 to each component of the unique solution of (5.10).

If one attempts to reduce (5.10) to a Cauchy problem by a change of independent variable $t \mapsto T - t$ in (5.10.b), then the resultant system will contain terms $u_1(t)$ and $u_2(T - t)$ and thereby be *non-local* in time.

PROPOSITION 5.5. *Let the spaces V_1, V_2, H , convex functions φ_1, φ_2 , operators $\mathcal{A}_1, \mathcal{A}_2$, and numbers $a, b \geq 0$ be given as in Proposition 5.4. Then for each pair $[f_1, f_2]$ and $[u_1^0, u_2^0]$ with $f_j \in L^{p_j}(0, T; V_j')$, $u_j^0 \in H$, $j = 1, 2$, there exists a unique solution of the system*

$$(5.11.a) \quad \begin{aligned} u_1 &\in L^{p_1}(0, T; V_1) : u_1' + \mathcal{A}_1(u_1) + au_2 + b(u_1 - u_2) = f_1 \\ &\text{in } L^{p_1}(0, T; V_1'), \end{aligned}$$

$$(5.11.b) \quad \begin{aligned} u_2 &\in L^{p_2}(0, T; V_2) : u_2' + \mathcal{A}_2(u_2) - au_1 + b(u_2 - u_1) = f_2 \\ &\text{in } L^{p_2}(0, T; V_2'), \end{aligned}$$

$$(5.11.c) \quad u_1(0) = u_1^0, \quad u_2(0) = u_2^0 \text{ in } H.$$

If in addition $f_j \in L^2(0, T; H)$, $u_j^0 \in V_j$ for $j = 1, 2$, then $u_j \in W^{1,2}(0, T; H)$ for $j = 1, 2$.

Systems of order greater than two can be resolved similarly and the operators may be time-dependent as in Proposition 5.1. Moreover, systems can be resolved with other constraints (such as periodic-in-time) or with singular coefficients as in (5.10).

Finally, each of the Propositions in this section applies directly to a corresponding parabolic partial differential equation or a system of such equations subject to boundary conditions and additionally to an evolution constraint such as Cauchy, periodic, singular or “reverse Cauchy” type. See Section 4 for typical examples and the following Example 6.A for time-dependent examples.

III.6. Degenerate Parabolic Equations

The technique of the preceding section leads to existence of solutions of *implicit evolution equations* which are nonlinear versions of those considered in Section 3. After giving an abstract form of the existence theorem which can be applied to resolve equations with Cauchy or periodic constraints, singular equations, and systems, we then briefly describe some model initial-boundary-value problems which contain partial derivatives of mixed elliptic-parabolic type.

PROPOSITION 6.1. *Let $\mathcal{V}$ be a separable reflexive Banach space and $\{S(t) : t \geq 0\}$ a contraction semigroup on $\mathcal{V}'$ with generator $-L$. Suppose that $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}'$ is type M , bounded and coercive. Let $\mathcal{B} : \mathcal{V} \rightarrow \mathcal{V}'$ be continuous and linear, and suppose we have*

$$S(t)\mathcal{B}v(v) \leq \mathcal{B}v(v) , \quad v \in \mathcal{V} , t \geq 0 .$$

Then for each $f \in \mathcal{V}'$ there exists a

$$(6.1) \quad u \in \mathcal{V} : L\mathcal{B}u + \mathcal{A}(u) = f \text{ in } \mathcal{V}' .$$

There is at most one such solution if $\mathcal{A}$ is strictly-monotone.

PROOF. The operator $(I - S(t))\mathcal{B}$ is monotone and weakly continuous, hence of type M , so by Example II.2.C it follows that $\frac{1}{\varepsilon}(I - S(\varepsilon))\mathcal{B} + \mathcal{A}$ is type M . This sum is bounded and coercive, so for each $\varepsilon > 0$ there exists a

$$(6.2) \quad u_\varepsilon \in \mathcal{V} : \frac{1}{\varepsilon}(I - S(\varepsilon))\mathcal{B}u_\varepsilon + \mathcal{A}(u_\varepsilon) = f \text{ in } \mathcal{V}'$$

by Theorem II.2.1. We see that $\mathcal{A}u_\varepsilon(u_\varepsilon) \leq f(u_\varepsilon)$, and so by passing to a subsequence (denoted by the same) we obtain

$$u_\varepsilon \rightharpoonup u \text{ in } \mathcal{V} , \quad \mathcal{A}u_\varepsilon \rightharpoonup g , \quad \mathcal{B}u_\varepsilon \rightharpoonup \mathcal{B}u \text{ in } \mathcal{V}' .$$

For any $v \in \mathcal{V}$ we have

$$(f - \mathcal{A}u_\varepsilon)(v) = \frac{1}{\varepsilon}(I - S(\varepsilon))\mathcal{B}u_\varepsilon(v) = \mathcal{B}u_\varepsilon\left(\frac{1}{\varepsilon}(I - S(\varepsilon))^*v\right) ,$$

so if $v \in D(L^*)$ it follows that

$$(f - g)(v) = \mathcal{B}u(L^*v) .$$

If $w \in \mathcal{V}$ then $v = (I + \varepsilon L^*)^{-1}w \in D(L^*)$, so we obtain

$$(f - g)(I + \varepsilon L^*)^{-1}w = L(I + \varepsilon L)^{-1}\mathcal{B}u(w) , \quad w \in \mathcal{V} .$$

This shows with II.1.2 that $\{L(I + \varepsilon L)^{-1}\mathcal{B}u : \varepsilon > 0\}$ is bounded in $\mathcal{V}'$, so $\mathcal{B}u \in D(L)$ and

$$L\mathcal{B}u + g = f .$$

Finally we note that

$$\begin{aligned} \mathcal{A}u_\varepsilon(u_\varepsilon - u) &= \left\langle f - \frac{1}{\varepsilon}(I - S(\varepsilon))\mathcal{B}u - \frac{1}{\varepsilon}(I - S_\varepsilon)\mathcal{B}(u_\varepsilon - u) , u_\varepsilon - u \right\rangle \\ &\leq \left\langle f - \frac{1}{\varepsilon}(I - S(\varepsilon))\mathcal{B}u , u_\varepsilon - u \right\rangle . \end{aligned}$$

Since $\mathcal{B}(u) \in D(L)$ the last term converges and we obtain $\limsup \mathcal{A}u_\varepsilon(u_\varepsilon - u) \leq 0$. But $\mathcal{A}$ is type M so $\mathcal{A}u = g$ and (6.1) holds. $\square$

We obtain a regularity result with respect to $\mathcal{B}$ as follows. Suppose $\mathcal{B}$ is also symmetric and monotone; let $\mathcal{V}_b$ be the seminorm space $\mathcal{V}$, $\langle \mathcal{B}\cdot, \cdot \rangle^{1/2}$. The dual $\mathcal{V}'_b$ is a Hilbert space with $\mathcal{V}'_b \subset \mathcal{V}'$. If $f \in \mathcal{V}'$, then $f \in \mathcal{V}'_b$ if and only if $|f(v)| \leq \text{const. } \mathcal{B}v(v)^{1/2}$, $v \in \mathcal{V}$. From the Cauchy-Schwartz inequality,

$$|\mathcal{B}w(v)| \leq \mathcal{B}w(w)^{1/2}\mathcal{B}v(v)^{1/2} ,$$

it follows that $\mathcal{B}w \in \mathcal{V}'_b$ for each $w \in \mathcal{V}$. It is clear that $\|\mathcal{B}w\|_{\mathcal{V}'_b} = \|w\|_{\mathcal{V}_b}$ in that case.

COROLLARY 6.1. *In addition to the assumptions of Proposition 6.1, suppose that $\mathcal{B}$ is symmetric and monotone, $\{S(t) : t \geq 0\}$ maps $\mathcal{V}$ into $\mathcal{V}$ and commutes with $\mathcal{B}$. Also assume*

$$\mathcal{A}v(S(t)v) \leq \mathcal{A}v(v) , \quad v \in \mathcal{V} , t \geq 0 .$$

Then for each $f \in \mathcal{V}'_b$ there exists a solution of (6.1) with $\mathcal{A}(u) \in \mathcal{V}'_b$.

PROOF. Apply (6.2) to $\frac{1}{\varepsilon}(I - S(\varepsilon))u_\varepsilon$. It follows that $\{\frac{1}{\varepsilon}(I - S(\varepsilon))\mathcal{B}u_\varepsilon\}$ is bounded in $\mathcal{V}'_b$, hence, $g \in \mathcal{V}'_b$. $\square$

Next we apply these abstract results to the *Cauchy Problem*. Let V be a separable reflexive Banach space and set $\mathcal{V} = L^p(0, T; V)$ with $p \geq 2$. Let $\mathcal{A}(t, \cdot) : V \rightarrow V'$ be given as in Proposition 5.1 for $0 \leq t \leq T$. Assume $\{\mathcal{B}(t) : 0 \leq t \leq T\}$ is a measurable family of symmetric monotone continuous linear operators from V to V' (see Section 2) and that

$$\begin{aligned} \int_0^T \|\mathcal{B}(t)\|_{L(V, V')}^q dt &< \infty , \quad q = p/(p-2) , \\ \mathcal{B}(s)v(v) &\leq \mathcal{B}(t)v(v) , \quad 0 \leq s < t \leq T , v \in V . \end{aligned}$$

PROPOSITION 6.2. *With the preceding assumptions, for each $f \in L^{p'}(0, T; V')$ there exists a*

$$(6.3) \quad u \in L^p(0, T; V) : \frac{d}{dt}(\mathcal{B}(t)u(t)) + \mathcal{A}(t, u(t)) = f(t) \quad \text{in } V'$$

at a.e. $t \in [0, T]$, and

$$\lim_{t \rightarrow 0} (\mathcal{B}(t)u(t)) = 0 \quad \text{in } V' .$$

PROOF. This will follow from Proposition 6.1 when we choose the semigroup as in Proposition 5.1. Note then the estimates

$$\begin{aligned} S(s)\mathcal{B}v(v) &= \int_s^T \mathcal{B}(t-s)v(t-s)(v(t)) dt \\ &\leq \int_s^T \mathcal{B}(t-s)v(t-s)(v(t-s))^{1/2} \mathcal{B}(t-s)v(t)(v(t))^{1/2} dt \\ &\leq \mathcal{B}v(v)^{1/2} \int_s^T \mathcal{B}(t)v(t)(v(t)) dt^{1/2} \leq \mathcal{B}v(v) , \quad v \in \mathcal{V} , 0 < s < T . \quad \square \end{aligned}$$

COROLLARY 6.2. *In addition, assume the following:*

$$(6.4.a) \quad \mathcal{B}(t) = \mathcal{B} \text{ is independent of } t ;$$

$$(6.4.b) \quad \mathcal{A}(t, v) = \varphi'(t, v) \text{ for each } t \in [0, T] ,$$

where the indicated G -differential is with respect to v and the convex functions $\varphi(t)$ are given as in Corollary 5.2 with estimates uniform in t ;

$$(6.4.c) \quad \varphi(s, v) \geq \varphi(t, v) , \quad 0 \leq s \leq t \leq T , v \in V .$$

Then for each $f \in L^{p'}(0, T; V'_b)$ there is a solution u of (6.3) with each term in $L^{p'}(0, T; V'_b)$.

PROOF. The essential point is to check that the subgradient inequality gives

$$\mathcal{A}(t, v(t))(S(s)v(t) - v(t)) \leq \varphi(t, S(s)v(t)) - \varphi(t, v(t)) , \quad \text{a.e. } t \in [0, T] ,$$

and then from the condition (6.4.c) on $\varphi(t, v)$ that

$$\int_0^T \varphi(t, S(s)v(t)) dt = \int_s^T \varphi(t, v(t-s)) dt \leq \int_0^{T-s} \varphi(t, v(t)) dt . \quad \square$$

The preceding resolves the equation (6.3) subject to homogeneous Cauchy data only. Somewhat more generally, if $\mathcal{B}(t)$ is a regular family as in Section 3 and if $\int_0^T \|\mathcal{B}'(t)\|_{L(V, V')}^{p'} dt < \infty$, then for $w_0 \in V$ we have $\frac{d}{dt}(\mathcal{B}(t)w_0) = \mathcal{B}'(t)w_0$ in $L^{p'}(0, T; V')$. If we solve for

$$w \in \mathcal{V} : \frac{d}{dt}(\mathcal{B}(t)w(t)) + \mathcal{A}(t, w(t) + w_0) = f(t) - \mathcal{B}'(t)w_0$$

and $\mathcal{B}w(0) = 0$ by Proposition 6.2, and then set $u(t) = w(t) + w_0$, we obtain a solution of (6.3) with $\mathcal{B}(0)u(0) = \mathcal{B}(0)w_0$ in V' . Our next objective is to find the optimal set of *initial data* for the Cauchy problem; this will be carried out in the special case of operators independent of time.

Assume B is continuous, linear, symmetric and monotone from the reflexive, separable Banach space V to its dual V' . Then $\langle B\cdot, \cdot \rangle^{1/2}$ is a seminorm on V ; denote the completion of this seminorm space by V_b and the dual Hilbert space by V'_b . Note that $V \hookrightarrow V_b$ is dense and continuous,

$$\|\varphi\|_{V_b} = B\varphi(\varphi)^{1/2} \leq \|B\|^{1/2}\|\varphi\| , \quad \varphi \in V ,$$

and B has a unique continuous linear extension from V_b onto V'_b . By restriction of functionals we have $V'_b \subset V'$, and this imbedding is continuous.

Define a seminorm on the range of $B : V \rightarrow V'$ by

$$\|w\|_W = \inf\{\|v\| : v \in V , Bv = w\} , \quad w \in R_g(B) .$$

Since the kernel $\ker(B)$ is closed this is a norm; the corresponding normed linear space $W = \{Rg(B), \|\cdot\|_W\}$ is isomorphic to the quotient $V/\ker(B)$, and it is therefore a Banach space. Finally, we observe that $W \subset V'_b$ with a continuous inclusion. Specifically, if $w = Bv$ with $v \in V$, then

$$|w(\varphi)| = |Bv(\varphi)| \leq Bv(v)^{1/2} B\varphi(\varphi)^{1/2} , \quad \varphi \in V$$

so $w \in V'_b$ and $\|w\|_{V'_b} = Bv(v)^{1/2} = \|v\|_{V_b} \leq \|B\|^{1/2}\|v\|$. Taking the infimum and noting B is constant on each coset, we obtain

$$\|w\|_{V'_b} \leq \|B\|^{1/2}\|w\|_W , \quad w \in Rg(B) .$$

The strict homomorphism $B : V \rightarrow W$ has a continuous dual $B' : W' \rightarrow V'$ given by

$$B'g(v) = g(Bv) , \quad g \in W' , v \in V .$$

Using the identification of $V_b \subset V''_b \subset W'$ we obtain for each $g \in V_b$

$$B'g(v) = Bg(v) \leq \|g\|_{V_b}\|v\|_{V_b} , \quad v \in V ,$$

so $\|B'g\|_{V'_b} \leq \|g\|_{V_b}$. B' is an extension of $B : V_b \rightarrow V'_b$ and hereafter we denote it too by B . Now V'_b is a Hilbert space whose scalar product satisfies

$$(Bu, Bv)_{V'_b} = Bu(v) = (u, v)_{V_b}, \quad u, v \in V_b.$$

Hence, $(f, w)_{V'_b} = f(v)$ if $w = Bv$, $v \in V_b$, so we obtain for each $f \in V'_b$

$$\begin{aligned} \sup\{|(f, w)_{V'_b}| : w \in W, \|w\|_W \leq 1\} = \\ \sup\{|f(v)| : v \in V, \|v\| \leq 1\} = \|f\|_{V'}. \end{aligned}$$

This shows V' has the norm *dual* to W with respect to V'_b , or that V'_b is the *pivot* space between W and V' .

PROPOSITION 6.3. *If $u \in \mathcal{V} = L^p(0, T; V)$ and $\frac{d}{dt}(Bu) \in \mathcal{V}'$, then $Bu \in C([0, T], V'_b)$. Conversely, for each $w \in V'_b$ there exists a $u \in \mathcal{V}$ with $\frac{d}{dt}(Bu) \in \mathcal{V}'$ and $\lim_{t \rightarrow 0^+} Bu(t) = w$ in V'_b .*

PROOF. From the preceding discussion we find

$$|(f, g)_{V'_b}| \leq \|f\|_{V'} \|g\|_W, \quad f \in V'_b, \quad g \in W$$

so the first part follows from the proof of Proposition 1.2. For the second part, we let $u_0 \in V_b$ with $Bu_0 = w$ and seek $u \in \mathcal{V}$ and $g \in \mathcal{V}'$ such that

$$\frac{d}{dt}(Bu) + g = 0, \quad Bu(0) = Bu_0.$$

Let $J : V \rightarrow V'$ be the duality map with gauge $\varphi(r) = r^{p-1}$ (see Section II.8) and $\{u_0^n\}$ a sequence in V with $u_0^n \rightarrow u_0$ in V_b . For each $n \geq 1$ there is a solution of

$$\begin{aligned} \frac{d}{dt}(Bv_n(t)) + J(v_n(t) + u_0^n) &= 0 \quad \text{in } \mathcal{V}' \\ Bv_n(0) &= 0 \end{aligned}$$

by Proposition 6.2. Set $u_n(t) = v_n(t) + u_0^n$, $0 \leq t \leq T$, and note that

$$\int_0^T J(u_n)(u_n) dt = - \int_0^T \frac{d}{dt} Bu_n(u_n) dt \leq \frac{1}{2} Bu_0^n(u_0^n)$$

by Proposition 1.2. By passing to a subsequence, denoted again by $\{u_n\}$, we have

$$u_n \rightharpoonup u \text{ in } \mathcal{V}, \quad J(u_n) \rightharpoonup g \text{ in } \mathcal{V}'.$$

For each $v \in W^{1,2}(0, T; V)$ with $v(T) = 0$ we obtain

$$- \int_0^T Bu_n(v') dt + \int_0^T J(u_n)(v) dt = Bu_0^n(v(0))$$

and then after letting $n \rightarrow \infty$ we obtain

$$- \int_0^T Bu(v') dt + g(v) = Bu_0(v(0)).$$

This implies $\frac{d}{dt}(Bu) = -g$ and by Proposition 1.2 that $Bu(0) = Bu_0$. □

COROLLARY 6.3. *In the situation of Proposition 6.2 with $\mathcal{B}(t) = \mathcal{B}$ (independent of t), for each $f \in L^{p'}(0, T; V')$ and $w_0 \in V'_b$, there exists a*

$$(6.5.a) \quad u \in L^p(0, T; V) : \frac{d}{dt}(\mathcal{B}u(t)) + \mathcal{A}(t, u(t)) = f(t) \quad \text{in } \mathcal{V}'$$

at a.e. $t \in [0, T]$ and

$$(6.5.b) \quad \lim_{t \rightarrow 0} \mathcal{B}u(t) = w_0 \quad \text{in } V'_b.$$

REMARK. Our attention has been focused on the Cauchy problem for (6.3). It is clear that by considering other semigroups as in Section 5 we can likewise resolve periodic or singular problems and various systems which contain similar equations. These developments are routine and we leave them for an exercise.

Examples of initial-boundary-value problems will be constructed. We do not seek the most general hypotheses but present a variety of relatively simple examples to illustrate some of the many possible applications of the abstract results.

EXAMPLE 6.A. QUASILINEAR PARABOLIC EQUATION. Let G be a bounded domain in $\mathbb{R}^n$, ∂G a C^1 manifold, and $2 \leq p < \infty$. Suppose that for each integer j , $0 \leq j \leq n$, we are given a function $a_j : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$(6.6.a) \quad a_j(t, \xi) \text{ is measurable in } t \text{ and continuous in } \xi,$$

$$(6.6.b) \quad |a_j(t, \xi)| \leq c|\xi|^{p-1} + k(t), \quad \xi \in \mathbb{R}, \quad 0 \leq t \leq T,$$

$$(6.6.c) \quad (a_j(t, \xi) - a_j(t, \eta))(\xi - \eta) \geq 0 \quad \text{for } \xi, \eta \in \mathbb{R},$$

$$(6.6.d) \quad a_j(t, \xi)\xi \geq \alpha|\xi|^p - k(t)$$

where $c_0 > 0$ and $k \in L^{p'}(0, T)$. Define

$$\begin{aligned} \mathcal{A}(t, u)(v) &= \int_G \sum_{j=1}^n a_j(t, \partial_j u(x)) \partial_j v(x) dx \\ &\quad + \int_{\partial G} a_0(t, \gamma u(s)) \gamma v(s) ds, \quad u, v \in W^{1,p}(G). \end{aligned}$$

Set $V = \{v \in W^{1,p}(G) : \gamma v = 0 \text{ a.e. on } \Gamma_0\}$ where Γ_0 is a measurable subset of ∂G . Then $\mathcal{A}$ satisfies the conditions in Proposition 5.1. Let $F \in L^{p'}(G \times (0, T))$, $g \in L^{p'}(\Gamma_1 \times (0, T))$, where $\Gamma_1 = \partial G \sim \Gamma_0$, and set

$$f(t)(v) = \int_G F(x, t)v(x) dx + \int_{\Gamma_1} g(s, t)\gamma v(s) ds, \quad v \in V,$$

for $0 \leq t \leq T$. Finally let $u_0 \in H = L^2(G)$. Then Proposition 5.1 asserts the existence of a unique generalized solution $u \in L^p(0, T; W^{1,p}(G))$ of the parabolic

initial-boundary-value problem

$$(6.7.a) \quad \frac{\partial u(x, t)}{\partial t} - \sum_{j=1}^n \partial_j a_j(t, \partial_j u(x, t)) = F(x, t), \quad x \in G,$$

$$(6.7.b) \quad u(s, t) = 0, \quad s \in \Gamma_0,$$

$$(6.7.c) \quad \sum_{j=1}^n a_j(t, \partial_j u(s, t)) \nu_j(s) + a_0(t, \gamma u(s)) = g(s, t), \quad s \in \Gamma_1,$$

for $t > 0$, and

$$(6.7.d) \quad u(x, 0) = u_0(x), \quad x \in G.$$

For each j let $\Psi_j(t, \eta) = \int_0^\eta a_j(t, \xi) d\xi$ and define the convex function

$$\varphi(t, v) = \int_G \sum_{j=1}^n \Psi_j(t, \partial_j v(x)) dx + \int_{\Gamma_1} \Psi_0(t, \gamma v(s)) ds, \quad v \in V.$$

Thus $\mathcal{A}(t, v) = \varphi_v(t, v)$ is a G -differential. Assume each $\Psi_j(t, \eta)$ is *non-increasing* in t . Then as above it follows that Corollary 5.1 applies. Thus, if $F \in L^2(G \times (0, T))$, $g \equiv 0$ and $u_0 \in V$, there is a solution of (6.7) with $\frac{\partial u}{\partial t} \in L^2(G \times (0, T))$. Similar examples can be given for the remaining abstract results of Section 5.

EXAMPLE 6.B. QUASILINEAR ELLIPTIC-PARABOLIC EQUATION. In order to obtain some typical examples of (6.3), suppose we are given a pair of functions $b \in L^q(0, T; L^{q^*}(G))$, $\beta \in L^q(0, T; L^q(\Gamma_1))$ such that

$$0 \leq b(\tau, x) \leq b(t, x) \text{ a.e. } x \in G, \quad 0 \leq \beta(\tau, s) \leq \beta(t, s) \text{ a.e. } s \in \Gamma_1, \\ \text{for a.e. } 0 \leq \tau \leq t \leq T,$$

where $q = p/(p-2)$, $q^* = p^*/(p^*-2)$ and the Sobolev imbedding $W^{1,p}(G) \hookrightarrow L^{p^*}(G)$ is continuous. Then the family of linear operators defined by

$$\mathcal{B}(t)u(v) = \int_G b(t, x)u(x)v(x) dx + \int_{\Gamma_1} \beta(t, s)\gamma u(s)\gamma v(s) ds, \quad u, v \in V, \quad 0 \leq t \leq T,$$

satisfies the hypotheses of Proposition 6.2. With the data as given it follows that there exists a generalized solution of the problem

$$(6.8.a) \quad \frac{\partial}{\partial t}(b(t, x)u) - \sum_{j=1}^n \partial_j a_j(t, \partial_j u) = F(x, t), \quad x \in G,$$

$$(6.8.b) \quad u(s, t) = 0, \quad s \in \Gamma_0,$$

$$(6.8.c) \quad \frac{\partial}{\partial t}(\beta(t, s)u(s, t)) + \sum_{j=1}^n a_j(t, \partial_j u)\nu_j(s) + a_0(t, u(s, t)) = g(s, t), \quad s \in \Gamma_1,$$

for $t > 0$ and

$$(6.8.d) \quad \lim_{t \rightarrow 0} (b(t, x)u(x, t)) = 0, \quad x \in G, \quad \lim_{t \rightarrow 0} \beta(t, s)u(s, t) = 0, \quad s \in \Gamma_1.$$

The partial differential equation (6.8.a) may be of elliptic or parabolic type, depending on whether or not $b(t, x) = 0$ at a specific time and position. Similarly, the boundary condition (6.8.c) will involve a time-derivative of the solution where

$\beta(t, s) > 0$, a *fourth type* boundary condition called a *dynamic boundary condition*. Otherwise it is either the classical *second type* or *third type*, and (6.8.b) is a boundary condition of *first type*.

Consider the autonomous case $b(t, x) = b(x)$, $\beta(t, s) = \beta(s)$ where we assume

$$0 \leq b(x) , \quad b \in L^q(G) , \quad 0 \leq \beta(s) , \quad \beta \in L^q(\Gamma_1) .$$

Thus we define $\mathcal{B} \in L(V, V')$ by

$$\mathcal{B}u(v) = \int_G b(x)u(x)v(x) dx + \int_{\Gamma_1} \beta(s)\gamma u(s)\gamma v(s) ds , \quad u, v \in V .$$

LEMMA. $V_b \cong b^{-1/2}L^2(G) \times \beta^{-1/2}L^2(\Gamma_1)$, where, e.g., we choose a representative $v \in b^{-1/2}L^2(G)$ with $b^{1/2}v \in L^2(G)$ and $v(x) = 0$ if $b(x) = 0$ for each $x \in G$, and $V'_b = b^{1/2}L^2(G) \oplus \beta^{1/2}L^2(\Gamma_1)$.

PROOF. The first part follows in part from the identity $\|v\|_{V_b}^2 = \|b^{1/2}v\|_{L^2(G)}^2 + \|\beta^{1/2}\gamma v\|_{L^2(\Gamma_1)}^2$. To get the second part, suppose first that $f \in V'$ and there is a constant $c > 0$ for which

$$|f(v)| \leq c \left(\int_G bv^2 dx \right)^{1/2} , \quad v \in V .$$

Letting $G_b = \{x \in G : b(x) > 0\}$, we have after extending f appropriately

$$|f(b^{-1/2}w)| \leq c \|w\|_{L^2(G_b)} , \quad w \in L^2(G_b) ,$$

so there is an $F \in L^2(G_b)$ such that

$$f(b^{-1/2}w) = (F, w)_{L^2(G_b)} , \quad w \in L^2(G_b) .$$

Thus, we obtain $f = b^{1/2}F$ on G_b and then extend as zero to all of G . Any part of f on Γ_1 is computed likewise. $\square$

If we use this Lemma with Corollary 6.3 we obtain the following. Assume $F \in L^{p'}(G \times (0, T))$ and $g \in L^{p'}(\Gamma_1 \times (0, T))$ and define $f(t)$ as before. Let $w_0 \in V'_b$ be given in the form $w_0 = bu_0 \oplus \beta v_0$ where $b^{1/2}u_0 \in L^2(G)$ and $\beta^{1/2}v_0 \in L^2(\Gamma_1)$. That is,

$$w_0(v) = \int_G bu_0 v dx + \int_{\Gamma_1} \beta v_0 (\gamma v) ds , \quad v \in V .$$

Then there is a generalized solution of

$$(6.9.a) \quad \frac{\partial}{\partial t}(b(x)u(x, t)) - \sum_{j=1}^n \partial_j a_j(t, \partial_j u(x, t)) = F(x, t) , \quad x \in G ,$$

$$(6.9.b) \quad u(s, t) = 0 , \quad s \in \Gamma_0 ,$$

$$(6.9.c) \quad \frac{\partial}{\partial t}(\beta(s)u(s, t)) + \sum_{j=1}^n a_j(t, \partial_j u)\nu_j(s) + a_0(t, u(s, t)) = g(s, t) , \quad s \in \Gamma_1 ,$$

for $t > 0$ and

$$(6.9.d) \quad \lim_{t \rightarrow 0} (b(x)u(x, t)) = b(x)u_0(x) , \quad x \in G$$

$$(6.9.e) \quad \lim_{t \rightarrow 0} \beta(s)u(s, t) = \beta(s)v_0(s) , \quad s \in \Gamma_1 .$$

The pair of initial conditions (6.9.d) and (6.9.e) hold in $b^{1/2}L^2(G)$ and $\beta^{1/2}L^2(\Gamma_1)$, respectively. That is, we have

$$\lim_{t \rightarrow 0} (b^{1/2}u(t)) = b^{1/2}u_0, \quad \lim_{t \rightarrow 0} (\beta^{1/2}\gamma u(t)) = \beta^{1/2}v_0$$

in $L^2(G)$ and $L^2(\Gamma_1)$, respectively. Moreover, we are in the situation of Corollary 6.2 if $F \in L^{p'}(0, T; b^{1/2}L^2(G))$, $g \in L^{p'}(0, T; \beta^{1/2}L^2(\Gamma_2))$ and $u_0 \in V$ with $v_0 = \gamma(u_0)$, hence $w_0 = \mathcal{B}u_0$. Then equation (6.9.a) holds in $L^{p'}(0, T; b^{1/2}L^2(G))$ and (6.9.c) holds in $L^{p'}(0, T; \beta^{1/2}L^2(\Gamma_1))$. Note that in order to obtain the more regular solution given by Corollary 6.2, we required the data F , g , u_0 and v_0 to be more regular and also the pair of initial data u_0 , v_0 to be *compatible*, i.e., $v_0 = \gamma u_0$. Finally, we see that this situation is covered by the abstract Green's formula (II.5.7), and this gives a sense to the boundary operator in (6.9.c).

EXAMPLE 6.C. POROUS MEDIUM EQUATION. Let G be a bounded domain in $\mathbb{R}^n$ and assume $2n/(n+2) \leq p < \infty$. Suppose that we are given a function $a : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\begin{aligned} a(t, \xi) &\text{ is measurable in } t \text{ and continuous in } \xi, \\ |a(t, \xi)| &\leq c|\xi|^{p-1} + k(t), \quad \xi \in \mathbb{R}, \quad 0 \leq t \leq T, \\ (a(t, \xi) - a(t, \eta))(\xi - \eta) &\geq 0 \quad \text{for } \xi, \eta \in \mathbb{R}, \\ a(t, \xi)\xi &\geq \alpha|\xi|^p - k(t), \end{aligned}$$

where $c_0 > 0$ and $k \in L^{p'}(0, T)$.

Consider the semilinear *porous medium equation* with Dirichlet boundary conditions,

$$(6.10.a) \quad \frac{\partial u(x, t)}{\partial t} - \Delta a(t, u(x, t)) = F(x, t), \quad x \text{ in } G,$$

$$(6.10.b) \quad a(t, u(s, t)) = 0, \quad s \text{ on } \partial G,$$

for $t \in (0, T]$. Recall that $H_0^1(G)$ is a Hilbert space with the scalar product

$$(\varphi, \psi)_{H_0^1} = \int_G \nabla \varphi \cdot \nabla \psi \, dx$$

and its dual is the space $H^{-1} = H_0^1(G)'$. The Riesz map $\mathcal{R} : H_0^1 \rightarrow H^{-1}$ is the isomorphism defined by $\mathcal{R}\varphi(\psi) = (\varphi, \psi)_{H_0^1}$, so we have $\mathcal{R} = -\Delta$. Also, we have the scalar product on H^{-1} given by

$$(f, g)_{H^{-1}} \equiv (\mathcal{R}^{-1}f, \mathcal{R}^{-1}g)_{H_0^1}, \quad f, g \in H^{-1},$$

and it satisfies the identities

$$\begin{aligned} (f, g)_{H^{-1}} &= \langle f, \mathcal{R}^{-1}g \rangle \\ &= \langle \mathcal{R}^{-1}f, g \rangle, \quad f, g \in H^{-1}. \end{aligned}$$

Take the scalar product of the (6.10.a) in the space H^{-1} to obtain

$$(u_t, g)_{H^{-1}} + (-\Delta a(t, u), g)_{H^{-1}} = (F, g)_{H^{-1}}, \quad g \in H^{-1}.$$

From the identities above we find the second term is just $\langle a(t, u), \varphi \rangle$ when we restrict it to $L^p(G)$, so we have

$$(u_t(t), \varphi)_{H^{-1}} + \int_G a(t, u(t)) \varphi \, dx = \int_G (-\Delta)^{-1} F(t) \varphi \, dx, \quad \varphi \in L^p(G).$$

This calculation shows how to realize the porous medium equation (6.10) as the abstract evolution equation (6.3).

Define $V = L^p(G)$ and

$$\mathcal{A}(t, u)(v) = \int_G a(t, u(x)) v(x) \, dx, \quad u, v \in L^p(G).$$

Let $\Psi(t, \eta) = \int_0^\eta a(t, \xi) d\xi$ and define the convex function

$$\varphi(t, v) = \int_G \Psi(t, v(x)) \, dx, \quad v \in V.$$

Thus $\mathcal{A}(t, v) = \varphi_v(t, v)$ is a G -differential. Assume $\Psi(t, \eta)$ is *non-increasing* in t . Then the family of operators $\mathcal{A}(t) : V \rightarrow V'$ satisfies the conditions in Corollary 6.2. Define $V_b = H^{-1}$ and

$$\mathcal{B}u(v) = (u, v)_{H^{-1}}, \quad u, v \in H^{-1}.$$

Then we are in the situation of Proposition 6.2 if we have the continuous imbedding $V \hookrightarrow V_b$, i.e., if $V_b' = H_0^1(G) \hookrightarrow L^{p'}(G) = V'$. By the Sobolev imbedding Theorem II.4.3 it is sufficient for this to have $p' \leq 2n/(n-2)$, i.e., to assume $p \geq 2n/(n+2)$. Let $F \in L^{p'}(0, T; L^{p'}(G))$, and set

$$f(t)(v) = \int_G F(x, t) v(x) \, dx, \quad v \in V,$$

for $0 \leq t \leq T$. From Proposition 6.2 it follows that there is a unique solution $u \in L^p(0, T; L^p(G))$ of (6.3) with each term in $L^{p'}(0, T; L^{p'}(G))$, i.e.,

$$\left\langle \frac{du(t)}{dt}, (-\Delta)^{-1} g \right\rangle + \langle a(t, u), g \rangle = \langle f(t), g \rangle, \quad g \in L^p(G).$$

This is equivalent to

$$\begin{aligned} (6.11) \quad u &\in L^p(0, T; L^p(G)) : \int_G \left(\frac{\partial u}{\partial t} \right) \varphi \, dx + \int_G a(t, u) (-\Delta) \varphi \, dx \\ &= \int_G F(\cdot, t) (-\Delta) \varphi \, dx, \quad \varphi \in H_0^1 : \Delta \varphi \in L^p, \end{aligned}$$

and so it gives a *very weak* notion of a solution of (6.10) with $\lim_{t \rightarrow 0} (-\Delta)^{-1} u(t) = 0$ in $L^{p'}(G)$. If we assume instead that $F \in L^{p'}(0, T; H^{-1})$ and define $f \in L^{p'}(0, T; H_0^1)$ by

$$f(t)(v) = \int_G ((-\Delta)^{-1} F)(x, t) v(x) \, dx, \quad v \in V,$$

then from Corollary 6.2 and Proposition 6.3 we obtain for each $u_0 \in H^{-1}$ a solution of

$$(6.12.a) \quad \frac{\partial u(t)}{\partial t} - \Delta a(t, u(t)) = F(t) \text{ in } H^{-1}(G),$$

$$(6.12.b) \quad a(t, u(t)) \in H_0^1(G) \text{ for a.e. } t \in (0, T),$$

$$(6.12.c) \quad \lim_{t \rightarrow 0} u(t) = u_0 \text{ in } H^{-1}(G),$$

with each term of (6.12.a) in $L^{p'}(0, T; H^{-1})$. In particular, it follows that we have $a(u(\cdot)) \in L^{p'}(0, T; H_0^1(G))$, so $a(u)$ satisfies the homogeneous Dirichlet boundary condition (6.12.b) as desired. This is a much more useful notion of solution, and in particular it is regular enough that the boundary condition (6.10.b) is meaningful in the sense of trace. We will refer to (6.12) as the H^{-1} -solution of (6.10).

III.7. Variational Inequalities

Let G be a bounded domain in $\mathbb{R}^n$ with smooth boundary ∂G . We showed in Section II.6 that for an appropriate function F on G there is a unique solution (in the corresponding space) of

$$(7.1.a) \quad -\Delta u(x) = F(x), \quad x \in G,$$

$$(7.1.b) \quad u \geq 0, \quad \frac{\partial u}{\partial \nu} \geq 0, \quad u \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial G.$$

This is an example of a *variational inequality of stationary type*. The nonlinear boundary conditions arose because we obtained the solution by minimizing the quadratic function

$$\frac{1}{2} \int_G |\nabla v|^2 dx - \int_G F(x)v(x) dx$$

over the *convex* (but non-linear) set $K = \{v \in H^1(G) : \gamma v \geq 0 \text{ in } L^2(\partial G)\}$. A natural problem suggested by the case of equations is to seek a solution of the variational inequality of *evolution type*

$$(7.2.a) \quad \frac{\partial u}{\partial t} - \Delta u = F(x, t), \quad x \in G, \quad t > 0,$$

$$(7.2.b) \quad u \geq 0, \quad \frac{\partial u}{\partial \nu} \geq 0, \quad u \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial G \times (0, \infty),$$

$$(7.2.c) \quad u(x, 0) = u_0(x), \quad x \in G$$

in which the elliptic equation (7.1.a) has been replaced by the corresponding parabolic equation (7.2.a). If u is a solution of (7.2) in the context of a model of heat conduction, then $\frac{\partial u}{\partial \nu}$ is that *control* of heat flux on the boundary which forces the temperature u to be non-negative on the boundary. The boundary constraints (7.2.b) show that the desired state has been obtained, that this state is obtained by a source on the boundary, and that the value of this source is zero at each point of ∂G where the temperature is strictly positive, respectively.

Our plan in this section is to show that *parabolic variational inequalities* such as (7.2) are well-posed. Suppose V is a Banach space and $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}'$, $\mathcal{V} = L^p(0, T; V)$,

$1 < p < \infty$. Let $\mathcal{K}$ be a convex subset of $\mathcal{V}$. Then we consider the *strong problem*

$$(7.3.a) \quad u \in \mathcal{K} : u' \in \mathcal{V}' ,$$

$$(7.3.b) \quad \int_0^T u'(v - u) dt + \mathcal{A}u(v - u) \geq f(v - u) , v \in \mathcal{K} ,$$

$$(7.3.c) \quad u(0) = u_0 ,$$

where $f \in \mathcal{V}'$ and $u_0 \in H$ are given. Since we have an inequality instead of an equation in (7.3.b) it is more difficult to obtain a solution with derivative as required by (7.3.a). It will be useful to consider a more general notion. Observe that if u is a strong solution of (7.3) and if $v \in \mathcal{K}$ with $v' \in \mathcal{V}'$, then

$$\begin{aligned} \int_0^T v'(v - u) dt + \mathcal{A}u(v - u) - f(v - u) &\geq \int_0^T (v' - u')(v - u) dt \\ &= (1/2)|v(T) - u(T)|^2 - 1/2|v(0) - u_0|^2 . \end{aligned}$$

Thus the function u satisfies the *weak problem*

$$(7.4) \quad \begin{aligned} u \in \mathcal{K} : \int_0^T v'(v - u) dt + \mathcal{A}u(v - u) &\geq f(v - u) \\ \text{for all } v \in \mathcal{K} \text{ with } v' \in \mathcal{V}' , v(0) &= u_0 . \end{aligned}$$

Note that the problem (7.4) requires only that $u \in L^p(0, T; V)$ with respect to regularity; there is no mention of u' . With reasonable conditions on $\mathcal{A}$ we shall demonstrate uniqueness of a weak solution of (7.4); then the existence of a strong solution corresponds to a *regularity* result. The stationary variational inequalities were resolved in Theorem II.2.3 for the “optimal” class of pseudo-monotone operators; the same will be done here for evolution problems of weak type (7.4). Strong solutions will be obtained for the special case of operators $\mathcal{A}$ which are linear, coercive on a Hilbert space $\mathcal{V}$ and compatible with $\mathcal{K}$.

We shall formulate both the weak and the strong problems in the abstract setting of Section 5. Thus, let $\mathcal{V}$ be a separable reflexive Banach space dense and continuous in a Hilbert space $\mathcal{H}$; identify $\mathcal{H} = \mathcal{H}'$ so we have $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$. Suppose $\{S(t) : t \geq 0\}$ is a contraction semigroup on $\mathcal{V}'$ and on $\mathcal{H}$. Denote its generators on $\mathcal{V}'$ and on $\mathcal{H}$ by $-L$ and $-L_{\mathcal{H}}$, respectively. As in Section 5, we shall regard L as an operator from $\mathcal{V}$ to $\mathcal{V}'$ with domain $\mathcal{V} \cap D(L)$. Let $\mathcal{K}$ be a closed, convex and non-empty subset of $\mathcal{V}$. The following condition of *compatibility* of L and $\mathcal{K}$ will be used.

LEMMA 7.1. *If $\mathcal{K}$ is invariant under the semigroup, i.e., $\{S(t)\}$ is a contraction semigroup on $\mathcal{V}$ and*

$$(7.5) \quad S(t)\mathcal{K} \subset \mathcal{K} , \quad t \geq 0 ,$$

then for every $v \in \mathcal{K}$ the sequence $\{v_\alpha\}$ defined by $v_\alpha + \alpha L v_\alpha = v$, $\alpha > 0$, satisfies

$$(7.6) \quad v_\alpha \in \mathcal{K} \cap D(L, \mathcal{V}') , \quad \lim_{\alpha \rightarrow 0} v_\alpha = v \text{ in } \mathcal{V} , \quad L v_\alpha (v_\alpha - v) \leq 0 .$$

PROOF. For each $\alpha > 0$ the resolvent of $-L$ is given by

$$(I + \alpha L)^{-1} + \int_0^\infty \frac{1}{\alpha} \exp(-t/\alpha) S(t) dt$$

and we note that $\int_0^\infty \frac{1}{\alpha} \exp(-t/\alpha) dt = 1$. Since $\mathcal{K}$ is closed and convex, (7.5) implies $(I + \alpha L)^{-1}\mathcal{K} \subset \mathcal{K}$. Thus, $v \in \mathcal{K}$ implies $v_\alpha \in \mathcal{K}$ and it is clear that $v_\alpha \rightarrow v$ in $\mathcal{V}$. Finally we note that $Lv_\alpha(v_\alpha - v) = -\alpha|Lv_\alpha|_{\mathcal{H}}^2$ so (7.6) follows. $\square$

The *existence* of weak solutions will be established for *pseudo-monotone* operators. Recall that the operator $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}'$ is pseudo-monotone if $u_n \rightharpoonup u$ and $\limsup \mathcal{A}(u_n)(u_n - u) \leq 0$ imply $\mathcal{A}u(u - v) \leq \liminf \mathcal{A}u_n(u_n - v)$ for $v \in \mathcal{V}$.

THEOREM 7.1 (BREZIS). *Assume the spaces $\mathcal{V}$, $\mathcal{H}$, the semigroup with generator $-L$, and the set $\mathcal{K}$ are given as above, and that the compatibility condition (7.6) holds. Let $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}'$ be bounded and pseudo-monotone, and assume that for some $v_0 \in \mathcal{K} \cap D(L)$ we have*

$$(7.7) \quad \frac{\mathcal{A}(v)(v - v_0)}{\|v\|} \rightarrow +\infty, \quad \text{as } \|v\| \rightarrow +\infty.$$

Then for every $f \in \mathcal{V}'$ there exists a

$$(7.8) \quad u \in \mathcal{K} : \langle Lv + \mathcal{A}(u) - f, v - u \rangle \geq 0, \quad v \in \mathcal{K} \cap D(L).$$

PROOF. Let $\varepsilon > 0$ and consider $\mathcal{B} : \mathcal{V} \rightarrow \mathcal{V}'$ defined by $\mathcal{B} = \frac{1}{\varepsilon}(I - S(\varepsilon)) + \mathcal{A}$. Since $S(\varepsilon)$ is a contraction on $\mathcal{H}$, and since $\frac{1}{\varepsilon}(I - S(\varepsilon))v_0$ is bounded independent of ε , it follows that

$$\begin{aligned} \mathcal{B}(v)(v - v_0) &= \frac{1}{\varepsilon}(I - S(\varepsilon))(v - v_0)(v - v_0) + \mathcal{A}v(v - v_0) + \frac{1}{\varepsilon}(I - S(\varepsilon))v_0(v - v_0) \\ &\geq \mathcal{A}v(v - v_0) - (\text{const.})\|v - v_0\|, \end{aligned}$$

so we find that

$$\frac{\mathcal{B}v(v - v_0)}{\|v\|} \rightarrow +\infty \quad \text{as } \|v\| \rightarrow +\infty.$$

From Theorem II.2.3 we obtain the existence of a sequence $\{u_\varepsilon\}$ such that

$$u_\varepsilon \in \mathcal{K} : \left(\frac{1}{\varepsilon}(I - S(\varepsilon))u_\varepsilon + \mathcal{A}(u_\varepsilon) - f \right)(v - u_\varepsilon) \geq 0 \quad \text{for } v \in \mathcal{K},$$

and from the coercivity estimate above that $\{u_\varepsilon\}$ is bounded in $\mathcal{V}$. By passing to a subsequence, which we denote again by $\{u_\varepsilon\}$, we have

$$u_\varepsilon \rightharpoonup u \text{ in } \mathcal{V}, u \in \mathcal{K}, \mathcal{A}u_\varepsilon \rightharpoonup g \text{ in } \mathcal{V}'.$$

Since each $I - S(\varepsilon)$ is monotone it follows that

$$(7.9) \quad \frac{1}{\varepsilon}(I - S(\varepsilon))v(v - u_\varepsilon) + \mathcal{A}u_\varepsilon(v - u_\varepsilon) \geq f(v - u_\varepsilon), \quad v \in \mathcal{K}.$$

If we further require that $v \in \mathcal{K} \cap D(L)$ then

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{A}u_\varepsilon(u_\varepsilon - u) \leq (Lv + g - f)(v - u).$$

By choosing $v = u_\alpha = (I + \alpha L)^{-1}u$ by (7.6) in this inequality and then letting $\alpha \rightarrow 0$ we obtain a lim sup on the right which is non-positive, so

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{A}u_\varepsilon(u_\varepsilon - u) \leq 0 .$$

Now $\mathcal{A}$ is pseudo-monotone so

$$\mathcal{A}u(u - v) \leq \liminf \mathcal{A}u_\varepsilon(u_\varepsilon - v) , \quad v \in \mathcal{V} .$$

From (7.9) we obtain then for each $v \in \mathcal{K} \cap D(L)$

$$\begin{aligned} \mathcal{A}u(u - v) &\leq \limsup \mathcal{A}u_\varepsilon(u_\varepsilon - v) \\ &\leq (Lv - f)(v - u) . \end{aligned}$$

That is, u is a (weak) solution of (7.8). $\square$

Next we give a sufficient condition for *uniqueness* of this solution.

PROPOSITION 7.1. *In addition to the assumptions of Theorem 7.1, suppose that $\mathcal{A}$ is strictly-monotone on $\mathcal{K}$. Then (7.8) has exactly one solution.*

PROOF. If u_1 and u_2 are solutions, then for each $v \in \mathcal{K} \cap D(L)$

$$\begin{aligned} \mathcal{A}u_1(v - u_1) + (Lv - f)(v - u_1) &\geq 0 , \\ \mathcal{A}u_2(v - u_2) + (Lv - f)(v - u_2) &\geq 0 . \end{aligned}$$

Set $u = \frac{1}{2}(u_1 + u_2)$ so $u \in \mathcal{K}$ and then define $u_\alpha = (I + \alpha L)^{-1}u$ for $\alpha > 0$. If we take $v = u_\alpha$ in the inequalities above and add, there follows

$$\begin{aligned} \mathcal{A}u_1(u_\alpha - u_1) + \mathcal{A}u_2(u_\alpha - u_2) &\geq \\ -2Lu_\alpha(u_\alpha - u) + 2f(u_\alpha - u) &\geq 2f(u_\alpha - u) . \end{aligned}$$

Taking the limit as $\alpha \rightarrow 0$ in the resulting inequality yields

$$\mathcal{A}u_1(u - u_1) + \mathcal{A}u_2(u - u_2) \geq 0 ,$$

so we obtain

$$\frac{1}{2}(\mathcal{A}u_1 - \mathcal{A}u_2)(u_1 - u_2) \leq 0 .$$

Since $\mathcal{A}$ is strictly-monotone, $u_1 = u_2$. $\square$

The following *regularity* result shows the solution is “strong” when the operator $\mathcal{A}$ is linear and the parabolic problem “ $L + \mathcal{A}$ ” is regularizing with respect to $\mathcal{K}$.

PROPOSITION 7.2. *In addition to the hypotheses of Theorem 7.1, assume $f \in \mathcal{H}$, $\mathcal{A}$ is linear and coercive from $\mathcal{V}$ to $\mathcal{V}'$, and*

$$(7.10) \quad \begin{cases} \text{for each } u \in \mathcal{K} \text{ there exists a } g \in \mathcal{H} \text{ such that} \\ \text{for every } \varepsilon > 0 \text{ there exists } u_\varepsilon \in \mathcal{K} \cap D(L) : \\ \varepsilon(Lu_\varepsilon + \mathcal{A}u_\varepsilon) + u_\varepsilon = u + \varepsilon g . \end{cases}$$

Then the solution of (7.8) satisfies

$$(7.11) \quad u \in \mathcal{K} \cap D(L) : (Lu + \mathcal{A}u - f)(v - u) \geq 0 , \quad v \in \mathcal{K} ,$$

and $Lu + \mathcal{A}(u) \in \mathcal{H}$.

PROOF. Let u be the weak solution of (7.8) obtained in Theorem 7.1; it is unique. It follows that

$$(Lv + \mathcal{A}v - f)(v - u) \geq 0, \quad v \in \mathcal{K} \cap D(L)$$

because $\mathcal{A}$ is monotone. Set $v = u_\varepsilon$ where u_ε is chosen according to (7.10). Thus we obtain

$$\varepsilon \langle Lu_\varepsilon + \mathcal{A}u_\varepsilon - f, g - Lu_\varepsilon + \mathcal{A}u_\varepsilon \rangle \geq 0$$

and then

$$|Lu_\varepsilon + \mathcal{A}u_\varepsilon|_{\mathcal{H}}^2 \leq (f + g, Lu_\varepsilon + \mathcal{A}u_\varepsilon)_{\mathcal{H}} - (f, g)_{\mathcal{H}}, \quad \varepsilon > 0.$$

Since $\{Lu_\varepsilon + \mathcal{A}u_\varepsilon\}$ is bounded in $\mathcal{H}$, (7.10) shows that $|u_\varepsilon - u|_{\mathcal{H}} = \mathcal{O}(\varepsilon)$.

Consider the extensions of L and $\mathcal{A}$ to $\mathcal{H}$ by transposition. Recall L is a continuous linear operator from $D(L_{\mathcal{H}})$ into $\mathcal{H}$ when $D(L_{\mathcal{H}})$ has graph-norm. The same is true for $L_{\mathcal{H}}^*$ and taking the continuous dual of this operator yields a continuous extension of L from $\mathcal{H}$ to $D(L_{\mathcal{H}}^*)'$. Likewise, we consider the restriction $\mathcal{A}_{\mathcal{H}}$ of $\mathcal{A}$ to $D(\mathcal{A}_{\mathcal{H}}) = \{v \in \mathcal{V} : \mathcal{A}v \in \mathcal{H}\}$ and put the $\mathcal{H}$ graph-norm on its domain to get a bounded operator. Taking the continuous dual of the adjoint operator $\mathcal{A}_{\mathcal{H}}^* \in \mathcal{L}(D(\mathcal{A}_{\mathcal{H}}^*), \mathcal{H})$ gives a continuous extension of $\mathcal{A}$ from $\mathcal{H}$ to $D(\mathcal{A}_{\mathcal{H}}^*)'$. We summarize these by the following diagrams.

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{L} & D(L_{\mathcal{H}}^*)' \\ \cup & & \cup \\ D(L) \cap \mathcal{V} & \longrightarrow & \mathcal{V}' \\ \cup & & \cup \\ D(L_{\mathcal{H}}) & \longrightarrow & \mathcal{H} \end{array} \qquad \begin{array}{ccc} \mathcal{H} & \xrightarrow{\mathcal{A}} & D(\mathcal{A}_{\mathcal{H}}^*)' \\ \cup & & \cup \\ \mathcal{V} & \longrightarrow & \mathcal{V}' \\ \cup & & \cup \\ D(\mathcal{A}_{\mathcal{H}}) & \longrightarrow & \mathcal{H} \end{array}$$

From all of the above it follows that

$$\begin{cases} u_\varepsilon \rightarrow u \text{ in } \mathcal{H}, \quad Lu_\varepsilon \rightarrow Lu \text{ in } D(L_{\mathcal{H}}^*)', \\ \mathcal{A}u_\varepsilon \rightarrow \mathcal{A}u \text{ in } D(\mathcal{A}_{\mathcal{H}}^*)', \text{ and} \\ Lu_\varepsilon + \mathcal{A}u_\varepsilon \rightarrow Lu + \mathcal{A}u \text{ in } \mathcal{H}. \end{cases}$$

It follows that $Lu = (Lu + \mathcal{A}u) - \mathcal{A}u \in \mathcal{V}'$ so $u \in D(L)$. Furthermore, for any $w \in \mathcal{K} \cap D(L)$ and $0 < t < 1$ we have $v_t \equiv (1-t)u + tw \in \mathcal{K} \cap D(L)$; from (7.8) follows

$$\langle Lv_t + \mathcal{A}u - f, w - u \rangle \geq 0,$$

and then we let $t \rightarrow 0$ to obtain

$$\langle Lu + \mathcal{A}u - f, w - u \rangle \geq 0, \quad w \in \mathcal{K} \cap D(L).$$

Finally, it follows from Lemma 7.1 that $\mathcal{K} \cap D(L)$ is dense in $\mathcal{K}$ for the $\mathcal{V}$ -norm, so the above inequality holds for each $w \in \mathcal{K}$. This establishes (7.11) for the solution of (7.8). $\square$

REMARK. The condition (7.10) is an *invariance property* of solutions of the parabolic equation. If $g = 0$, it requires that if $u \in \mathcal{K}$ then $u_\varepsilon \in \mathcal{K}$, a generalized *maximum principle* if, e.g., $\mathcal{K}$ consists of non-negative functions. This is relaxed by permitting a *control*, g , as we shall see below.

The next regularity result holds for nonlinear $\mathcal{A}$ but requires the set $\mathcal{K}$ to be nearly invariant under the semigroup and its adjoint and that f be smooth in time.

PROPOSITION 7.3. *In addition to the hypotheses of Theorem 7.1, assume $f \in D(L)$ and that*

$$(7.12) \quad (\mathcal{A}(u) - \mathcal{A}(v))(u - v) \geq c\|u - v\|^2, \quad u, v \in \mathcal{V},$$

$$(7.13) \quad S(t)\mathcal{A}(v) = \mathcal{A}(S(t)v), \quad t \geq 0, v \in \mathcal{V},$$

and there exists a $\lambda > 0$ such that

$$(7.14) \quad S(t)v + S(t)^*v - S(t)^*S(t)v + (\lambda - 1)v \in \lambda\mathcal{K}, \quad v \in \mathcal{K}, t \geq 0.$$

Then the solution of (7.8) is strong in the sense of (7.11) and, moreover, $Lu \in \mathcal{V}$.

PROOF. Let u_ε be the solutions of the difference-approximations as in the proof of Theorem 7.1; multiply by λ to obtain

$$\left(\frac{1}{\varepsilon}(I - S(\varepsilon))u_\varepsilon + \mathcal{A}(u_\varepsilon) - f\right)(\lambda v - \lambda u_\varepsilon) \geq 0, \quad v \in \mathcal{K}.$$

Since $u_\varepsilon \in \mathcal{K}$ we can use (7.14) to choose $v \in \mathcal{K}$ by

$$\begin{aligned} v &= u_\varepsilon - \frac{1}{\lambda}(I - S(t)^*)(I - S(t))u_\varepsilon \\ &= \frac{1}{\lambda}\{S(t)u_\varepsilon + S(t)^*u_\varepsilon - S(t)^*S(t)u_\varepsilon + (\lambda - 1)u_\varepsilon\} \end{aligned}$$

and thereby obtain

$$(I - S(t))\left(\frac{1}{\varepsilon}(I - S(\varepsilon))u_\varepsilon + \mathcal{A}(u_\varepsilon) - f\right)\left((I - S(t))u_\varepsilon\right) \leq 0.$$

From this inequality and (7.13) follows

$$\begin{aligned} \left(\mathcal{A}(u_\varepsilon) - \mathcal{A}(S(t)u_\varepsilon)\right)(u_\varepsilon - S(t)u_\varepsilon) + \frac{1}{\varepsilon}(I - S(\varepsilon))\left((I - S(t))u_\varepsilon\right)(I - S(t))u_\varepsilon \\ \leq (I - S(t))f\left((I - S(t))u_\varepsilon\right), \end{aligned}$$

and then (7.12) and monotonicity of $I - S(\varepsilon)$ imply further that

$$c\|(I - S(t))u_\varepsilon\|_{\mathcal{V}} \leq \|(I - S(t))f\|_{\mathcal{V}'}.$$

By weak-lower-semicontinuity, let $\varepsilon \rightarrow 0$ to get the estimate

$$c\|(I - S(t))u\|_{\mathcal{V}} \leq \|(I - S(t))f\|_{\mathcal{V}'}, \quad t \geq 0.$$

From the semigroup representation

$$\frac{1}{t}(I - S(t))u = L\left(\frac{1}{t} \int_0^t S(s)u \, ds\right),$$

the fact that L is closed, and the boundedness of the left side, it follows that $u \in D(L)$ and $Lu \in \mathcal{V}$. Thus (7.11) follows from the argument at the end of Proposition 7.2. $\square$

REMARK. Proposition 3 gives $Lu \in \mathcal{V}$, a regularity result much stronger than the desired condition $Lu \in \mathcal{V}'$ which is appropriate for the “strong” solution of (7.11).

The application of Theorem 7.1 to variational inequalities of evolution is straightforward. Consider the situation of Proposition 5.1 where the spaces $V \subset H \subset V'$, $\mathcal{V} = L^p(0, T; V)$ with $p \geq 2$, $\mathcal{H} = L^2(0, T; H)$, and operators $\mathcal{A}(t, \cdot) : V \rightarrow V'$ are given with $\mathcal{A}(t, v)$ being measurable in t , and monotone, hemicontinuous, bounded and coercive in v . The realization, $\mathcal{A}v(t) = \mathcal{A}(t, v(t))$, is monotone and hemicontinuous, hence, pseudomonotone on $\mathcal{V}$. Suppose that for each $t \in [0, T]$, $K(t)$ is a closed, convex subset of V and it contains 0. Then

$$\mathcal{K} = \{v \in \mathcal{V} : v(t) \in K(t) \text{ a.e. } t \in [0, T]\}$$

is likewise closed and convex in $\mathcal{V}$.

To determine the *Cauchy problem* in $\mathcal{V}$, let $\{S(t) : t \geq 0\}$ be the semigroup of translations used in Proposition 5.1, so the corresponding generator $-L$ is given by $Lu = u'$ with $u(0) = 0$. Assume the convex sets are non-decreasing:

$$(7.15) \quad K(s) \subset K(t) \text{ if } 0 \leq s \leq t \leq T.$$

Then for $v \in \mathcal{K}$ we have $S(t)v(s) \in K(s-t) \subset K(s)$ for $0 \leq s \leq T$, hence, $S(t)v \in \mathcal{K}$ for $t \geq 0$, and the compatibility condition (7.6) holds. From Theorem 7.1 it follows that for each $f \in \mathcal{V}$ there exists a *weak solution* of

$$(7.16) \quad u \in \mathcal{K} : \int_0^T \left(v'(t) + \mathcal{A}(t, u(t)) - f(t) \right) (v(t) - u(t)) dt \geq 0$$

for all $v \in \mathcal{K}$ with $v' \in \mathcal{V}'$ and $v(0) = 0$.

This is exactly (7.4) with $u_0 = 0$. Non-homogeneous initial conditions, $u(0) = u_0$, can be attained if $u_0 \in K$. However, the more natural condition $u_0 \in \overline{K}$, the H closure, seems more difficult. Furthermore, if a.e. $\mathcal{A}(t, \cdot)$ is strictly-monotone then uniqueness holds for (7.16).

Other evolution problems can be obtained likewise. The singular *history-value problem* of Proposition 5.3 can be resolved for variational inequalities in the situation above. Similarly the *periodic problem* has a weak solution, but the convex sets are limited to a constant $K(t) = K \subset V$ in order to attain (7.5) and compatibility with the periodic semigroup. Then we obtain existence of

$$(7.17) \quad u \in \mathcal{K} : \int_0^T \left(v'(t) + \mathcal{A}(t, u(t)) - f(t) \right) (v(t) - u(t)) dt \geq 0$$

for all $v \in \mathcal{K}$ with $v' \in \mathcal{V}'$ and $v(0) = v(T)$.

In like manner we obtain some interesting variational inequalities for systems; see Section 4 and below.

Examples of initial-boundary-value problems can be constructed as before. Thus, set $V = W^{1,p}(G)$ and define

$$\mathcal{A}(t, u)v = \int_G \sum_{j=1}^n a_j(t, \partial_j u) \partial_j v \, dx + \int_{\partial G} a_0(t, \gamma u) \gamma v \, ds, \quad u, v \in V,$$

where the functions $a_j : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy (6.6). Let $F \in L^{p'}(G \times (0, T))$, $g \in L^{p'}(\partial G \times (0, T))$ and set

$$f(t)v = \int_G F(x, t)v(x) \, dx + \int_{\partial G} g(s, t)\gamma v(s) \, ds, \quad v \in V.$$

These operators and functionals will be used to construct four examples.

EXAMPLE 7.A. BOUNDARY CONSTRAINT. For each $t \in [0, T]$ let $\Gamma(t)$ be a measurable subset of ∂G , and assume this family is non-increasing:

$$\Gamma(s) \supset \Gamma(t) \quad \text{if } 0 \leq s \leq t \leq T.$$

Define corresponding subsets of V by

$$K(t) = \{v \in W^{1,p}(G) : \gamma v(s) \geq 0, \text{ a.e. } s \in \Gamma(t)\}.$$

Then (7.15) holds and Theorem 7.1 asserts the existence of a *weak solution* of

$$(7.18.a) \quad \frac{\partial u}{\partial t} - \sum_{j=1}^n \partial_j a_j(t, \partial_j u) = F \quad \text{in } G$$

$$(7.18.b) \quad u \geq 0, \quad \partial_{A(t)}(u) - g \geq 0, \quad (\partial_{A(t)}(u) - g)u = 0 \quad \text{on } \Gamma(t),$$

$$(7.18.c) \quad \partial_{A(t)}(u) = g(\cdot, t) \quad \text{on } \partial G \sim \Gamma(t)$$

for each $t > 0$, and $u(\cdot, 0) = 0$ in G . Here we have used the abstract boundary operator $\partial_{A(t)}$ of Section II.5 which is given on smooth functions by

$$\partial_{A(t)}(u) = \sum_{j=1}^n a_j(t, \partial_j u) \nu_j + a_0(t, \gamma u)$$

on ∂G . Recall that $\vec{\nu}$ is the unit outward normal. Note that the parabolic partial differential equation (7.18.a) is to hold in G with the standard variational boundary condition (7.18.c) on a prescribed part of the boundary. The remaining part $\Gamma(t)$ consists of two parts, *not* known *a-priori*, the first where

$$u > 0, \quad \partial_{A(t)}(u) = g$$

and the second part where the constraint is attained,

$$u = 0, \quad \partial_{A(t)}(u) \geq g.$$

We emphasize that all of these characterizations of a weak solution are purely *formal*.

EXAMPLE 7.B. INTERIOR CONSTRAINT. In order to simplify (and ignore) boundary conditions in the following, we choose $V = W_0^{1,p}(G)$. Thus, we take hereafter $a_0 \equiv 0$ and $g \equiv 0$ in $\mathcal{A}$ and f above, respectively. Let $w \in W^{1,p}$ with $w \leq 0$ a.e. in G ; then define

$$K = \{v \in V : v(x) \geq w(x) \text{ a.e. } x \in G\}.$$

(Note that we could likewise let w depend on t so long as it is non-increasing with t .) From Theorem 7.1 follows the existence of a *weak* solution of

$$(7.19.a) \quad \begin{cases} \frac{\partial u}{\partial t} - \sum_{j=1}^n \partial_j a_j(t, \partial_j u) \geq F, & u \geq w, \\ \left(\frac{\partial u}{\partial t} - \sum_{j=1}^n \partial_j a_j(t, \partial_j u) - F \right) (u - w) = 0 & \text{in } G \times (0, T), \end{cases}$$

$$(7.19.b) \quad u = 0 \text{ on } \partial G \times (0, T),$$

with the initial condition $u(\cdot, 0) = 0$ in G . The region $G \times (0, T)$ is partitioned into two parts, one $\Omega^+ = \{(x, t) \in G \times (0, T) : u(x, t) > w(x)\}$ on which the equation (7.18.a) holds and the remainder on which $u = w$. Furthermore, one expects u and $\vec{\nabla} u$ to be continuous across the unknown free surface $\partial\Omega^+$.

EXAMPLE 7.C. GRADIENT CONSTRAINT. Let the space V and data $\mathcal{A}$, f be given as before but set

$$K = \{v \in V : |\vec{\nabla} v(x)| \leq 1 \text{ a.e. } x \in G\}.$$

This leads to a weak solution of

$$(7.20.a) \quad \frac{\partial u}{\partial t} - \sum_{j=1}^n \partial_j a_j(t, \partial_j u) = F \quad \text{where } |\vec{\nabla}_x u(x, t)| < 1,$$

$$(7.20.b) \quad |\vec{\nabla}_x u(x, t)| = 1 \quad \text{otherwise}$$

in $G \times (0, T)$ with u and $\vec{\nabla} u$ continuous across the free surface which separates these two regions, and the boundary condition

$$(7.20.c) \quad u = 0 \text{ on } \partial G \times (0, T)$$

and initial condition $u(\cdot, 0) = 0$ on G . Such problems arise as elastic-plastic torsion models where (7.20.a) is the *elastic* region and (7.10.b) is the *plastic* region.

Examples of parabolic systems of variational inequalities can be constructed easily. If K is a product of convex sets, one each from the component spaces V_1 , V_2 in the situation of Proposition 5.5, then we essentially obtain a coupled pair of variational inequalities of the preceding types. The following system is more interesting and serves as a model for diffusion in two layers separated by a semi-conductive barrier. This provides a standard resistance to one direction of flux but absolutely no flow in the reverse direction.

EXAMPLE 7.D. SYSTEM CONSTRAINT. Let $V_j = W^{1,p_j}(G)$, $p_j \geq 2$, and $\mathcal{A}_j(t, \cdot) : V_j \rightarrow V'_j$ be a partial differential operator constructed as above for $j = 1, 2$. Thus we have

$$\mathcal{A}_j(t, u) = - \sum_{i=1}^n \partial_i \alpha_i^j(t, \partial_i u) , \quad u \in V_j , \quad 0 \leq t \leq T .$$

Define $\mathcal{V} = L^{p_1}(0, T; V_1) \times L^{p_2}(0, T; V_2)$, $\mathcal{H} = L^2(0, T; L^2(G)) \times L^2(0, T; L^2(G))$ and $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}'$ by

$$\begin{aligned} \mathcal{A}(u)(v) &= \int_0^T \left(\mathcal{A}_1(t, u_1(t))(v_1(t)) + \mathcal{A}_2(t, u_2(t))(v_2(t)) \right) dt \\ &\quad + (au_2 + b(u_1 - u_2), v_1)_{L^2} + (-au_1 + b(u_2 - u_1), v_2)_{L^2} , \\ u &= [u_1, u_2] , \quad v = [v_1, v_2] \in \mathcal{V} , \end{aligned}$$

where $a, b \geq 0$. Then $\mathcal{A}$ is monotone, hemicontinuous, bounded and coercive. Let $\{S(t) : t \geq 0\}$ be the semigroup on $\mathcal{V}$, $\mathcal{H}$, $\mathcal{V}'$ of translations on respective components. (The corresponding evolution equation was described in Proposition 5.5.) Define

$$\mathcal{K} = \{v = [v_1, v_2] \in \mathcal{V} : v_1 \geq v_2 \text{ a.e. on } G \times (0, T)\} .$$

It follows easily that $\mathcal{K}$ and $\{S(t) : t \geq 0\}$ are compatible so Theorem 7.1 applies to this situation. To characterize the weak solution, note that if $f = [f_1, f_1] \in \mathbb{R}^2$ and if $C = \{v \in \mathbb{R}^2 : v_1 \geq v_2\}$, then

$$u \in C : f(v - u) \geq 0 , \quad v \in C$$

is equivalent to

$$u_1 \geq u_2 , \quad f_1 = -f_2 \geq 0 , \quad f_1(u_1 - u_2) = 0 .$$

Thus, for each $f = [f_1, f_1] \in \mathcal{V}'$, we obtain a *weak solution* of

$$\begin{aligned} (7.21.a) \quad \frac{\partial u_1}{\partial t} + \mathcal{A}_1(t, u_1) + au_2 + b(u_1 - u_2) - f_1 \\ = - \left(\frac{\partial u_2}{\partial t} + \mathcal{A}_2(t, u_2) - au_1 + b(u_2 - u_1) - f_2 \right) \geq 0 , \end{aligned}$$

$$(7.21.b) \quad u_1 \geq u_2 , \quad \text{and}$$

$$(7.21.c) \quad \left(\frac{\partial u_1}{\partial t} + \mathcal{A}_1(t, u_1) + au_2 + b(u_1 - u_2) - f_1 \right) (u_1 - u_2) = 0$$

in $G \times (0, T)$ together with

$$(7.21.d) \quad u_1 = u_2 = 0 \text{ on } \partial G \times (0, T) ,$$

$$(7.21.e) \quad u_1(\cdot, 0) = u_2(\cdot, 0) = 0 \text{ in } G .$$

The region $G \times (0, T)$ is thereby partitioned into two parts, the first on which $u_1 > u_2$ and the pair of equations (5.11.a), (5.11.b) hold, the second on which $u_1 = u_2$ and the non-negative quantity (7.21.a) is the distributed flux from the second component into the first component.

Finally, we shall show that the regularity results above apply to our first two examples.

EXAMPLE 7.A. BOUNDARY CONSTRAINT (CONTINUED). Assume further that $p = 2$ and the operator $\mathcal{A}$ is independent of time and the coefficients are strongly monotone. Then we obtain (7.12) and (7.13). Also let $\Gamma(t) = \Gamma$ be independent of time and note that for $v \in \mathcal{V}$ we have

$$v(s) - S(t)^* S(t)v(s) = \begin{cases} 0, & 0 \leq t \leq s \leq T - t, \\ v(s), & \text{otherwise,} \end{cases}$$

and hence (7.14) holds with $\lambda = 2$. Here $\mathcal{K}$ is a cone with vertex at the origin; this simplifies the calculation but it is not necessary for (7.14). Proposition 7.3 now implies that if $f \in D(L)$ then $u \in D(L)$ and $Lu \in \mathcal{V}$. That is, if $f \in W^1(0, T; L^2(G))$ and $f(x, 0) = 0$, $x \in G$, then we have a *strong solution* of (7.18), and we can then check that

$$\mathcal{A}(u) \in L^2(0, T; H^1(G)), \quad u \in W^{1,2}(0, T; H^1(G)).$$

Additional regularity of u may be obtained from the regularity theory for stationary variational inequalities for the Dirichlet-Laplace operator in $L^2(G)$.

EXAMPLE 7.B. INTERIOR CONSTRAINT (CONTINUED). Restrict further the operator $\mathcal{A}$ to be linear; for simplicity set

$$\mathcal{A}u(v) = \int_G \vec{\nabla} u \cdot \vec{\nabla} v \, dx = -\Delta u(v), \quad u, v \in H_0^1(G)$$

although any linear second-order elliptic operator will suffice. We shall verify that (7.10) holds when the constraint w belongs to $H^2(G)$. Now let $u \in \mathcal{K}$ and set $g = -\Delta w$. For $\varepsilon > 0$ there is a unique solution u^ε of the *linear* problem

$$\begin{aligned} \varepsilon \left(\frac{\partial u^\varepsilon}{\partial t} - \Delta u^\varepsilon \right) + u^\varepsilon &= u + \varepsilon g \quad \text{in } \mathcal{H} \\ u^\varepsilon &\in L^2(0, T; H_0^1(G)), \quad u^\varepsilon(0) = 0 \end{aligned}$$

and this satisfies

$$\varepsilon \left(\frac{\partial}{\partial t} (u^\varepsilon - w) - \Delta (u^\varepsilon - w) \right) + (u^\varepsilon - w) = (u - w)$$

with each term in $L^2(G \times (0, T))$. Since $(u^\varepsilon - w)^- \in L^2(0, T; H_0^1(G))$ we obtain

$$\begin{aligned} \varepsilon \frac{1}{2} \frac{d}{dt} \int_G ((u^\varepsilon - w)^-)^2 \, dx &+ \int_G |\vec{\nabla} (u^\varepsilon - w)^-|^2 \, dx \\ &+ \int_G ((u^\varepsilon - w)^-)^2 \, dx = \int_G (u - w)(u^\varepsilon - w)^- \, dx. \end{aligned}$$

The right side is non-positive since $u \in \mathcal{K}$ and it follows $(u^\varepsilon - w)^- = 0$, hence, $u^\varepsilon \in \mathcal{K}$. Proposition 7.2 can be applied: since $F \in \mathcal{H}$ the weak solution of (7.19) is actually a *strong solution* with $\frac{\partial u}{\partial t} - \Delta u \in \mathcal{H}$. From regularity theory of this parabolic equation it follows that

$$u \in L^2(0, T; H^2(G)), \quad u \in W^{1,2}(0, T; L^2(G)).$$

CHAPTER IV

Accretive Operators and Nonlinear Cauchy Problems

IV.1. Accretive Operators in Hilbert Space

We briefly review various special results on real-valued, convex, lower-semi-continuous functionals on a Hilbert space. All of these have been obtained in a more general setting in II.7, but we collect them here for convenience and give usually substantially more simple proofs in Hilbert space to make the presentation independent. The subgradient motivates the extension of the notion of m -accretive operators as given in I.4, and in the remainder of this section we develop the properties of these multi-valued operators.

Let H be a Hilbert space. A sequence $\{x_n\}$ in H is *weakly convergent* to $x \in H$ if $\lim_{n \rightarrow \infty} (x_n, y)_H = (x, y)_H$ for every $y \in H$. In view of the isomorphism between H and its dual, H' , this definition is consistent with weak convergence in a general Banach space, and we shall denote the weak convergence by " $x_n \rightharpoonup x$ ". The important property of weak sequential compactness for reflexive Banach spaces is particularly simple to verify for Hilbert space.

PROPOSITION 1.1. *If $\{x_n\}$ is a bounded sequence in the Hilbert space H , then it has a weakly convergent subsequence.*

PROOF. It suffices to consider the case in which H is *separable*, i.e., it has a countable dense subset $\{v_n\}$. (This is the case if H is the closed linear span of the sequence.) Then the general case is easily obtained from the projection Corollary I.2.2. Pick a subsequence $\{x_{1,n}\}$ of $\{x_n\}$ for which $\{(x_{1,n}, v_1)_H\}$ converges. Inductively, pick a subsequence $\{x_{j,n}\}$ of $\{x_{j-1,n}\}$ such that $\{(x_{j,n}, v_k)_H\}$ converges for $k = 1, 2, \dots, j$. Thus, $\{x_{n,n}\}$ is a subsequence of $\{x_n\}$ for which $\lim_{n \rightarrow \infty} (x_{n,n}, v_j)_H$ exists for all $j \geq 1$. Now define $f(y) = \lim_{n \rightarrow \infty} (x_{n,n}, y)_H$ for all y in $\langle v_j \rangle$, the linear span of $\{v_j\}$. Then f is linear and continuous, so it has a unique extension to all of H , again denoted by $f \in H'$. From Corollary I.3 it follows there is an $x \in H$ such that $\lim_{n \rightarrow \infty} (x_{n,n}, y)_H = f(y) = (x, y)_H$ for $y \in \langle v_j \rangle$, and from the boundedness of $\{x_{n,n}\}$ it follows that $x_{n,n} \rightharpoonup x$ as desired. $\square$

Since strong convergence of a sequence implies weak convergence, every weakly-closed set is (strongly) closed. (Note that sequences are adequate to describe the (strong) closure of a set.) The following Proposition shows that the weak and strong closure are identical for convex sets.

PROPOSITION 1.2. *If K is closed and convex in H then K is weakly closed.*

PROOF. Let x_0 be a point not in K ; we construct a weak-neighborhood separating them. Define $x = P_K(x_0)$, the indicated projection; we may assume

$(x + x_0)/2 = 0$. Then $(x - x_0, y - x)_H \geq 0$ for $y \in K$, so we have $(x, y)_H \geq \|x\|^2$ for $y \in K$, and $(x, x_0)_H < 0$. That is, the linear functional $\mathcal{R}_H x \in H'$ separates K and x_0 . $\square$

COROLLARY 1.1. *A convex set is weakly closed if and only if it contains all weak limits of sequences in K .*

Let the function $\varphi : H \rightarrow (-\infty, +\infty] \equiv \mathbb{R}_\infty$ be convex, i.e.,

$$\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y), \quad x, y \in H, \quad 0 \leq t \leq 1.$$

Then its domain $\text{dom}(\varphi) = \{x \in H : \varphi(x) < \infty\}$ is convex in H and each sublevel set $E_c = \{x \in H : \varphi(x) \leq c\}$ is convex. The function φ is *lower-semi-continuous* if each such E_c is closed, $c \in \mathbb{R}$. From Corollary 1.1 it follows that this is the same for weak and strongly closed and for weakly sequentially closed. Moreover we have the following criterion.

PROPOSITION 1.3. *The convex function $\varphi : H \rightarrow \mathbb{R}_\infty$ is lower-semi-continuous if and only if $\lim_{n \rightarrow \infty} x_n = x$ implies $\varphi(x) \leq \liminf_{n \rightarrow \infty} \varphi(x_n)$.*

PROOF. The limit condition implies each E_c is closed: $x_n \in E_c$ and $x_n \rightarrow x$ implies $\varphi(x) \leq c$, hence, $x \in E_c$. Conversely, suppose $\lim_{n \rightarrow \infty} x_n = x$ and that there is an $\varepsilon > 0$ for which

$$\varphi(x) - \varepsilon > b \equiv \liminf_{n \rightarrow \infty} \varphi(x_n).$$

There is a subsequence $\{x_{n_k}\}$ such that $\varphi(x_{n_k}) \leq b + \varepsilon/2$ and $E_{b+\varepsilon/2}$ is closed so $\varphi(x) \leq b + \varepsilon/2$, a contradiction. $\square$

PROPOSITION 1.4. *Let K be closed convex and non-empty in H , and let $\varphi : H \rightarrow (-\infty, +\infty]$ be convex and lower-semi-continuous. If*

$$(1.1) \quad \lim_{x \in K, \|x\| \rightarrow \infty} \varphi(x) = +\infty,$$

then there exists a minimum point

$$x_0 \in K : \varphi(x_0) \leq \varphi(y), \quad y \in K.$$

PROOF. Let $\{x_n\}$ be a minimizing sequence in K . Then (1.1) shows either φ is $+\infty$ on K , so any $x_0 \in K$ is a solution, or that $\{x_n\}$ is bounded. Proposition 1.1 shows some subsequence is weakly convergent, Proposition 1.2 that the limit x_0 belongs to K , and Proposition 1.3 that x_0 is a minimum point. $\square$

We recall that the function φ is *proper* if $\text{dom}(\varphi)$ is non-empty: $\varphi(x) < \infty$ for some $x \in H$. Assume that $\varphi : H \rightarrow \mathbb{R}_\infty$ is convex, lower-semi-continuous, and proper. This is equivalent to requiring that the *epigraph* $\text{epi}(\varphi) \equiv \{[x, t] \in H \times \mathbb{R} : \varphi(x) \leq t\}$ be, respectively, convex, closed, and non-empty.

LEMMA 1.1. *The function φ is lower-bounded by a continuous affine function.*

PROOF. Take any point $[x_0, r_0] \in H \times \mathbb{R}$ for which $x_0 \in \text{dom}(\varphi)$ and $[x_0, r_0] \notin \text{epi}(\varphi)$ and define $[x_1, r_1]$ to be the projection of $[x_0, r_0]$ on $\text{epi}(\varphi)$ in $H \times \mathbb{R}$. This is characterized by

$$[x_1, r_1] \in \text{epi}(\varphi) : ([x_1, r_1] - [x_0, r_0], [x, t] - [x_1, r_1])_{H \times \mathbb{R}} \geq 0, \quad [x, t] \in \text{epi}(\varphi).$$

That is, we have $\varphi(x_1) \leq r_1$ and

$$(1.2) \quad (x_1 - x_0, x - x_1)_H + (r_1 - r_0)(t - r_1) \geq 0 \quad \text{if} \quad \varphi(x) \leq t.$$

If we choose $x = x_1$ in this inequality, we find that $r_1 \geq r_0$. If $r_1 = r_0$ then we likewise obtain $(x_1 - x_0, x - x_1)_H \geq 0$ for all $x \in \text{dom}(\varphi)$, hence, $x_1 = x_0$, and this is a contradiction because $[x_0, r_0]$ does not belong to $\text{epi}(\varphi)$. It follows that $r_1 > r_0$. Setting $x = x_1$ and $t = \varphi(x_1)$ in (1.2) shows that $\varphi(x_1) \geq r_1$, hence, $\varphi(x_1) = r_1$. Set $w = (x_0 - x_1)/(r_1 - r_0)$. Since we may take $t = \varphi(x)$ in (1.2) we obtain

$$(1.3) \quad (w, x - x_1)_H \leq \varphi(x) - \varphi(x_1), \quad x \in H,$$

so the continuous affine function ℓ defined by

$$\ell(x) = (w, x - x_1) + \varphi(x_1)$$

is a lower bound on φ . □

Actually the function ℓ is *exact* at x_1 , $\ell(x_1) = \varphi(x_1)$, so it is a maximal linear lower bound on the epigraph of φ . These maximal lower bounds correspond to the subgradients of φ .

DEFINITION. Let $\varphi : H \rightarrow \mathbb{R}_\infty$ be convex, proper, and lower-semi-continuous. Then $w \in H$ is a *subgradient* of φ at $x_1 \in H$ if

$$(w, x - x_1)_H \leq \varphi(x) - \varphi(x_1), \quad x \in H.$$

The set of all subgradients of φ at x_1 is denoted by $\partial_H \varphi(x_1)$.

REMARKS. If we identify H with H' by the Riesz isomorphism, $\mathcal{R}_H$, then the subgradients agree with the subdifferentials (II.7). In general, however, the subgradient and the Riesz isomorphism depend explicitly on the scalar product in H . In particular we have

$$w \in \partial_H \varphi(x_1) \Leftrightarrow \mathcal{R}_H w \in \partial \varphi(x_1),$$

and this is denoted by $\mathcal{R}_H \circ \partial_H \varphi = \partial \varphi$.

The subgradient $\partial_H \varphi$ is a multi-valued operator or *relation* on $H \times H$. It is accretive (see I.4) in the following sense: if $w_j \in \partial_H \varphi(x_j)$ for $j = 1, 2$, then $(w_1 - w_2, x_1 - x_2)_H \geq 0$. To see this, note that

$$\begin{aligned} (w_1, x_2 - x_1)_H &\leq \varphi(x_2) - \varphi(x_1), \\ (w_2, x_1 - x_2)_H &\leq \varphi(x_1) - \varphi(x_2), \end{aligned}$$

and then add these inequalities. Moreover, we have the following.

PROPOSITION 1.5. *If $\varphi : H \rightarrow \mathbb{R}_\infty$ is convex, proper, and lower-semi-continuous, then the range of $I + \partial_H \varphi$ is all of H .*

PROOF. Let $w_0 \in H$ and define $\psi : H \rightarrow \mathbb{R}_\infty$ by

$$\psi(x) = \varphi(x) + 1/2\|x\|^2 - (w_0, x)_H, \quad x \in H.$$

Since φ has an affine lower bound, ψ satisfies $\lim_{\|x\| \rightarrow \infty} \psi(x) = +\infty$, so by Proposition 1.4 it attains a minimum value at some $x_0 \in H$: that is, $0 \in \partial_H \psi(x_0)$, or

$$(w_0, x - x_0)_H \leq \varphi(x) - \varphi(x_0) + 1/2(\|x\|^2 - \|x_0\|^2), \quad x \in H.$$

Replace x by $tx + (1-t)x_0$, use the convexity of φ , divide by $t > 0$, and let $t \rightarrow 0^+$ to obtain

$$(w_0, x - x_0)_H \leq \varphi(x) - \varphi(x_0) + (x_0, x - x_0)_H, \quad x \in H.$$

This shows $w_0 - x_0 \in \partial_H \varphi(x_0)$ as desired. $\square$

COROLLARY 1.2. (cf. Proposition I.4.2). For every $\alpha > 0$, $Rg(I + \alpha \partial_H \varphi) = H$.

The preceding results lead us to extend our notion of m -accretive operators to include possibly nonlinear or multi-valued relations. For any relation A on H , that is, a subset A of $H \times H$, we define the *domain* $D(A) = \{x : [x, y] \in A\}$, *range* $Rg(A) = \{y : [x, y] \in A\}$ and *inverse* $A^{-1} = \{[y, x] : [x, y] \in A\}$. This extends the notion of a function when we identify it with its graph. Conversely, we may regard A as a function into “subsets of H ” with the notation $A(x) = \{y : [x, y] \in A\}$; then A is a function exactly when each $A(x)$ is a single point, and $D(A) = \{x : A(x) \text{ is non-empty}\}$. The linear operations lead to the definitions

$$\lambda A = \{[x, \lambda y] : [x, y] \in A\} \quad \text{for } \lambda \in \mathbb{R} \quad \text{and}$$

$$A + B = \{[x, y + z] : [x, y] \in A \quad \text{and} \quad [x, z] \in B\}.$$

Then $D(\lambda A) = D(A)$ if $\lambda \neq 0$, and $D(A + B) = D(A) \cap D(B)$.

DEFINITION. The relation or operator A on H is *accretive* if $[x_j, w_j] \in A$ for $j = 1, 2$, (i.e., $w_j \in A(x_j)$) implies $(w_1 - w_2, x_1 - x_2)_H \geq 0$. It is *m -accretive* if, in addition, $Rg(I + A) = H$.

Such operators will play a central role in the following where we shall generalize and extend the results of I.5. We begin with some preliminary extensions of results from I.4 which characterize m -accretive operators.

LEMMA 1.2. $(x, y)_H \geq 0$ if and only if

$$\|x\| \leq \|x + \alpha y\|, \quad \alpha > 0.$$

PROOF. Both of these conditions are equivalent to

$$0 \leq 2\alpha(x, y)_H + \alpha^2\|y\|^2, \quad \alpha > 0. \quad \square$$

COROLLARY 1.3. The following are equivalent:

- (a) A is accretive,
- (b) $\|x_1 - x_2\| \leq \|(x_1 + \alpha w_1) - (x_2 + \alpha w_2)\|$ for all $[x_j, w_j] \in A$, $j = 1, 2$, and $\alpha > 0$, and
- (c) $(I + \alpha A)^{-1}$ is a contraction on $Rg(I + \alpha A)$ for all $\alpha > 0$.

LEMMA 1.3. *The following are equivalent:*

- (a) *A is accretive and $Rg(I + \alpha A) = H$ for some $\alpha > 0$,*
- (b) *A is m -accretive, and*
- (c) *A is accretive and $Rg(I + \alpha A) = H$ for all $\alpha > 0$.*

PROOF. It suffices to show (a) implies (c). Assume (a) and let $\beta > \alpha/2$. For a given $w \in H$ we define $T : H \rightarrow H$ by

$$T(x) = (I + \alpha A)^{-1} \left(\frac{\alpha}{\beta} w + \left(1 - \frac{\alpha}{\beta} \right) x \right), \quad x \in H.$$

Since $(I + \alpha A)^{-1}$ is a contraction and

$$\left| 1 - \frac{\alpha}{\beta} \right| = \frac{|\beta - \alpha|}{\beta} < 1,$$

it follows that T is a strict-contraction on H . Thus T has a unique fixed point, $x = T(x)$, which then satisfies $x + \beta A(x) \ni w$. This shows $Rg(I + \beta A) = H$ if $\beta > \alpha/2$, and (c) follows by an easy induction. $\square$

Let A be m -accretive and define the corresponding *resolvents* of A by $J_\alpha \equiv (I + \alpha A)^{-1}$ for $\alpha > 0$. Each J_α is a contraction on H . From the equivalence of $y = J_\alpha(x)$ and $\frac{1}{\alpha}(x - y) \in A(y)$, we obtain

$$(1.4) \quad \frac{1}{\alpha}(x - J_\alpha(x)) \in A(J_\alpha(x)), \quad x \in H, \quad \alpha > 0.$$

We can write (1.4) in the form

$$\frac{1}{\beta} \left(\frac{\beta}{\alpha} x + \left(1 - \frac{\beta}{\alpha} \right) J_\alpha(x) - J_\alpha(x) \right) \in A(J_\alpha(x)), \quad \beta > 0,$$

and this is equivalent to

$$\frac{\beta}{\alpha} x + \left(1 - \frac{\beta}{\alpha} \right) J_\alpha(x) \in (I + \beta A)(J_\alpha(x)).$$

This proves the *resolvent identity*

$$(1.5) \quad J_\alpha = J_\beta \circ \left(\frac{\beta}{\alpha} I + \left(1 - \frac{\beta}{\alpha} \right) J_\alpha \right), \quad \alpha, \beta > 0.$$

PROPOSITION 1.6. *Let A be m -accretive.*

- (a) *Then A is maximal accretive: if B is accretive and $A \subset B$, then $A = B$.*

Let $[x_n, y_n] \in A$ and $x_n \rightarrow x$, $y_n \rightarrow y$ in H .

- (b) *If $\liminf_{n \rightarrow \infty} (x_n, y_n)_H \leq (x, y)_H$ then $[x, y] \in A$.*
- (c) *If $\limsup_{n \rightarrow \infty} (x_n, y_n)_H \leq (x, y)_H$, then $\lim_{n \rightarrow \infty} (x_n, y_n)_H = (x, y)_H$.*

PROOF.

- (a) If $w \in B(x)$, let $z = (I + A)^{-1}(w + x)$. Then $z \in D(A) \subset D(B)$ and $w + x \in (I + B)(z) \cap (I + B)(x)$, so $x = z \in D(A)$ and $w \in A(x)$.
- (b) Taking the $\liminf$ of the inequality

$$(v - y_n, u - x_n)_H \geq 0, \quad [u, v] \in A, \quad n \geq 1,$$

gives

$$(v - y, u - x)_H \geq 0, \quad [u, v] \in A.$$

Since A is maximal accretive, it equals $B = A \cup \{[x, y]\}$.

- (c) From (b) we have $(y - y_n, x - x_n)_H \geq 0$, $n \geq 1$. Take the $\liminf$ and obtain $\liminf_{n \rightarrow \infty} (x_n, y_n)_H \geq (x, y)_H$, so (c) follows immediately. $\square$

Actually, every maximal accretive operator in Hilbert space is m -accretive. Proposition 1.6.a is the easier half of this deep result of G. Minty on the equivalence of these two notions. The remaining parts of Proposition 1.6 give continuity properties of m -accretive operators. With the identity (1.4) these will lead us next to results on the approximations of I by J_α and of A by m -accretive Lipschitz operators.

PROPOSITION 1.7 (MINTY-ROCKAFELLAR). *If A is m -accretive then $\overline{D(A)}$ is convex and $\lim_{\alpha \rightarrow 0} J_\alpha(x) = \text{Proj}_{\overline{D(A)}}(x)$ for each $x \in H$.*

PROOF. Let K be the closed convex hull of $D(A)$ and $x \in H$. Set $x_\alpha = J_\alpha(x)$ so $\frac{1}{\alpha}(x - x_\alpha) \in A(x_\alpha)$ by (1.4). Since A is accretive

$$\left(\frac{1}{\alpha}(x - x_\alpha) - v, x_\alpha - u \right)_H \geq 0, \quad [u, v] \in A,$$

so we obtain

$$\|x_\alpha\|^2 \leq (x, x_\alpha - u)_H + (x_\alpha, u)_H - \alpha(v, x_\alpha - u)_H, \quad \alpha > 0.$$

Thus $\{x_\alpha\}$ is bounded, so some subsequence $\{x_{\alpha'}\}$ converges weakly to a vector $x_0 \in K$. Taking the $\liminf$ in the above shows that

$$\|x_0\|^2 \leq (x, x_0 - u)_H + (x_0, u)_H, \quad u \in D(A),$$

so we have

$$x_0 \in K : (x - x_0, u - x_0)_H \geq 0, \quad u \in K,$$

and this shows $x_0 = \text{Proj}_K(x)$. By the uniqueness of weak limits we have $x_\alpha \rightharpoonup x_0$. From the estimate on $\|x_\alpha\|$ we also obtain

$$\limsup \|x_\alpha\|^2 \leq (x, x_0 - u)_H + (x_0, u)_H, \quad u \in K,$$

and setting $u = x_0$ shows $\limsup \|x_\alpha\|^2 \leq \|x_0\|^2$. Hence $\lim_{\alpha \rightarrow 0} \|x_\alpha\| = \|x_0\|$ so we have the strong $\lim_{\alpha \rightarrow 0} x_\alpha = x_0$ in H . Finally, note that if $x \in K$ then $x_\alpha \in D(A)$ and $x_\alpha \rightarrow x$ so it follows $\overline{D(A)} = K$. $\square$

Consider now the following approximation of the m -accretive operator A by a Lipschitz function. For $\alpha > 0$ given, add αI to the inverse A^{-1} and then invert this sum: $A_\alpha = (A^{-1} + \alpha I)^{-1}$. For operators on $H = \mathbb{R}$, A_α is obtained from A by tilting the graph to obtain a maximum slope of $1/\alpha$. In general, we have the equivalence of

$$\begin{aligned} y &\in (A^{-1} + \alpha I)^{-1}x, & x &\in (\alpha I + A^{-1})y, \\ y &\in A(x - \alpha y), & x &\in (I + \alpha A)(x - \alpha y), \end{aligned}$$

and $y = \frac{1}{\alpha}(x - J_\alpha(x))$. Compare this with (1.4).

DEFINITION. Let A be m -accretive on H . Then the *Yosida approximations* of A is the operator

$$(1.6) \quad A_\alpha \equiv \frac{1}{\alpha}(I - J_\alpha), \quad \alpha > 0.$$

Thus, $A_\alpha(x) \in A(J_\alpha(x))$, $x \in H$, and A_α is characterized by $y = A_\alpha(x)$ if and only if $y \in A(x - \alpha y)$. Since A is maximal accretive, we can show that each $A_\alpha(x)$ is closed and convex in H . This allows us to define the *minimal section operator* A^0 by

$$A^0x = \text{Proj}_{A(x)}(0) = \{y : y \in A(x), y \text{ of minimal norm}\}.$$

THEOREM 1.1. *Let A be m -accretive.*

- (a) *Each A_α is m -accretive and Lipschitz with constant $\frac{1}{\alpha}$, $\alpha > 0$.*
- (b) *$(A_\alpha)_\beta = A_{\alpha+\beta}$, $\alpha, \beta > 0$.*
- (c) *For each $x \in D(A)$, $\|A_\alpha x\|$ converges upward to $\|A^0x\|$, $\lim_{\alpha \rightarrow 0} A_\alpha(x) = A^0x$, and*

$$\|A_\alpha x - A^0x\|^2 \leq \|A^0x\|^2 - \|A_\alpha x\|^2, \quad \alpha > 0.$$

- (d) *For each $x \notin D(A)$, $\|A_\alpha x\|$ is increasing and unbounded as $\alpha \rightarrow 0$.*

PROOF.

- (a) From the definition of A_α and (1.4) we obtain

$$\begin{aligned} & (A_\alpha x_1 - A_\alpha x_2, x_1 - x_2)_H \\ &= (A_\alpha x_1 - A_\alpha x_2, \alpha(A_\alpha x_1 - A_\alpha x_2) + (J_\alpha x_1 - J_\alpha x_2))_H \\ &\geq \alpha \|A_\alpha x_1 - A_\alpha x_2\|^2 \end{aligned}$$

since A is accretive. Therefore A_α is accretive and satisfies

$$\|A_\alpha x_1 - A_\alpha x_2\| \leq \frac{1}{\alpha} \|x_1 - x_2\|, \quad x_1, x_2 \in H.$$

From the definition of A_α it follows that x satisfies the resolvent equation $w = (I + \beta A_\alpha)(x)$ if x is a fixed-point of the function $T(x) = w - \beta A_\alpha \text{ lpha}(x)$. Such a fixed-point exists if $\beta < \alpha$, for then T is a strict contraction.

- (b) This follows from the characterization of $y = A_\alpha x$ by $y \in A(x - \alpha y)$.
- (c) If $x \in D(A)$ then

$$0 \leq (A^0x - A_\alpha x, x - J_\alpha x)_H = \alpha(A^0x - A_\alpha x, A_\alpha x)_H$$

so we have $\|A_\alpha x\|^2 \leq (A^0x, A_\alpha x)_H$ and $\|A_\alpha x\| \leq \|A^0x\|$. Applied to A_β with (b) we obtain $\|A_{\alpha+\beta}x\|^2 \leq (A_\beta x, A_{\alpha+\beta}x)_H$ and $\|A_{\alpha+\beta}x\| \leq \|A_\beta x\|$, so it follows that $\|A_\alpha(x)\|$ increases as α decreases and satisfies

$$\|A_\alpha x - A_{\alpha+\beta}x\|^2 \leq \|A_\alpha x\|^2 - \|A_{\alpha+\beta}x\|^2, \quad \alpha, \beta > 0,$$

for each $x \in H$. If $\{\|A_\alpha x\|\}$ is bounded then this inequality shows that there is a $y \in H$ for which $\lim_{\alpha \rightarrow 0} A_\alpha(x) = y$. Also (1.6) shows $\lim_{\alpha \rightarrow 0} J_\alpha(x) = x$, and so from Proposition 6 and (1.4) we obtain $y \in A(x)$. Also $\|y\| = \lim_{\alpha \rightarrow 0} \|A_\alpha x\| \leq \|A^0x\|$, so $y = A^0x$. This also proves (d). $\square$

Finally we return to the special case of a subgradient, $A = \partial\varphi$, and show the approximations A_α are derivatives likewise.

DEFINITION. Let $f : H \rightarrow \mathbb{R}$ be a function and $x \in H$. Then a vector $f'(x) \in H$ is the Frechet gradient of f at x if

$$\lim_{y \rightarrow x} \left[\left(f(y) - f(x) - (f'(x), y - x)_H \right) / \|y - x\| \right] = 0$$

If such a $f'(x)$ exists we say f is Frechet differentiable at x .

PROPOSITION 1.8 (MOREAU). Let $\varphi : H \rightarrow \mathbb{R}_\infty$ be proper, convex and lower-semi-continuous; set $A = \partial\varphi$. Define for each $\alpha > 0$

$$\varphi_\alpha(x) \equiv \min \left\{ \frac{1}{2\alpha} \|y - x\|^2 + \varphi(y) : y \in H \right\}, \quad x \in H.$$

Then $\varphi_\alpha(x) = \frac{\alpha}{2} \|A_\alpha x\|^2 + \varphi(J_\alpha(x))$, and φ_α is convex and Frechet differentiable with gradient $\varphi'_\alpha = A_\alpha$. Also $\varphi_\alpha(x)$ converges upward to $\varphi(x)$ as $\alpha \downarrow 0$ for each $x \in H$.

PROOF. From the proof of Proposition 1.5 it follows that the minimum in the definition of φ_α is attained at y if and only if $0 \in \frac{1}{\alpha}(y - x) + \partial\varphi(y)$, and this is equivalent to $y = J_\alpha(x)$. This shows $\varphi_\alpha(x) = \frac{1}{2\alpha} \|x - J_\alpha x\|^2 + \varphi(J_\alpha(x))$ as desired.

Let $x, y \in H$; from $A_\alpha(x) \in \partial\varphi(J_\alpha x)$ it follows that $(A_\alpha x, J_\alpha y - J_\alpha x)_H \leq \varphi(J_\alpha y) - \varphi(J_\alpha x)$, hence,

$$\varphi_\alpha(y) - \varphi_\alpha(x) \geq \frac{\alpha}{2} \left\{ \|A_\alpha y\|^2 - \|A_\alpha x\|^2 + \frac{2}{\alpha} (A_\alpha x, J_\alpha y - J_\alpha x)_H \right\}.$$

Writing $J_\alpha = I - \alpha A_\alpha$ gives

$$\begin{aligned} \varphi_\alpha(y) - \varphi_\alpha(x) &\geq \frac{\alpha}{2} \left\{ \|A_\alpha y\|^2 - \|A_\alpha x\|^2 + 2(A_\alpha x, A_\alpha x - A_\alpha y)_H + \frac{2}{\alpha} (A_\alpha x, y - x)_H \right\} \\ &= \frac{\alpha}{2} \|A_\alpha y - A_\alpha x\|^2 + (A_\alpha x, y - x)_H \\ &\geq (A_\alpha x, y - x)_H. \end{aligned}$$

Interchanging x, y gives

$$\varphi_\alpha(x) - \varphi_\alpha(y) \geq (A_\alpha y, x - y)_H = (A_\alpha x, x - y)_H + (A_\alpha y - A_\alpha x, x - y)_H$$

so we obtain

$$0 \leq \varphi_\alpha(y) - \varphi_\alpha(x) - (A_\alpha x, y - x)_H \leq \frac{1}{\alpha} \|y - x\|^2.$$

This shows $\varphi'_\alpha(x) = A_\alpha(x)$. Since A_α is accretive, it follows by Proposition II.7.4 that φ_α is convex.

Now from the definition of φ_α we obtain

$$\varphi_\beta(x) \leq \varphi_\alpha(x) \leq \varphi(x), \quad 0 < \alpha < \beta$$

and we have seen above that

$$\varphi(J_\beta x) \leq \varphi_\beta(x).$$

By Proposition 1.7 it follows that for each $x \in \overline{D(A)}$, $\lim J_\alpha(x) = x$ and so

$$\varphi(x) \leq \liminf \varphi(J_\alpha x) \leq \liminf \varphi_\alpha(x) \leq \limsup \varphi_\alpha(x) \leq \varphi(x).$$

That is, $\lim \varphi_\alpha(x) = \varphi(x)$. But for each $x \notin \overline{D(A)}$, $\|x - J_\alpha x\| \rightarrow \|\text{Proj}_{\overline{D(A)}}(x) - x\| > 0$ and $\alpha \|A_\alpha x\|^2 = \|A_\alpha x\| \|x - J_\alpha x\| \rightarrow +\infty$, so

$$\varphi_\alpha(x) \rightarrow +\infty = \varphi(x) . \quad \square$$

COROLLARY 1.4. $\text{dom}(\varphi) \subset \overline{D(A)}$, hence, $D(A) \subset \text{dom}(\varphi) \subset \overline{\text{dom}(\varphi)} \subset \overline{D(A)}$.

IV.2. Construction of m -accretive Operators

We first recall some examples of m -accretive operators which are subgradients of convex functions as discussed in Section II.8. This class includes a variety of elliptic boundary-value problems. Then we introduce the realization of an m -accretive operator in the Hilbert space H that is distributed over $L^2(\Omega, H)$ and realizations of first-order derivatives of either real or vector-valued functions of several variables corresponding to initial-value problems. By adding such operators we shall obtain evolution equations and (as special cases) a variety of parabolic initial-boundary-value problems.

EXAMPLE 2.A: CONVEX FUNCTIONS ON $\mathbb{R}$.

Let $F : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ be a monotone function and denote its left and right limits at $x \in \mathbb{R}$ by $F^-(x)$ and $F^+(x)$, respectively. Then $F^-(x) = F^+(x)$ at all but at most a countable set of points. The subgradient of the convex function

$$\varphi(x) \equiv \int_{x_0}^x F^-(s) ds = \int_{x_0}^x F^+(s) ds$$

for a given $x_0 \in \text{dom}(F)$ is given by

$$\partial\varphi(x) = [F^-(x), F^+(x)] , \quad x \in \mathbb{R} .$$

EXAMPLE 2.B: CONVEX INTEGRANDS.

Let Ω be a measurable subset of $\mathbb{R}^n$ and $\varphi : \mathbb{R} \rightarrow \mathbb{R}_\infty$ be proper, convex, and lower-semi-continuous. We shall also suppose that either $0 = \varphi(0) = \min(\varphi)$. (Note that if Ω has finite measure, then we may use an affine exact lower bound for φ to change variable and thereby reduce to this case. That is, if $\varphi(x) \geq \ell(x)$ for the affine continuous function, $\ell(\cdot)$, then we replace $\varphi(\cdot)$ by $\varphi(\cdot) - \ell(\cdot)$.) Then

$$\Phi(u) \equiv \int_{\Omega} \varphi(u(x)) dx$$

defines a proper, convex, lower-semi-continuous function $\Phi : L^2(\Omega) \rightarrow \mathbb{R}_\infty^+$ with domain $\text{dom}(\Phi) = \{u \in L^2(\Omega) : \varphi \circ u \in L^1(\Omega)\}$, and the subgradient is computed pointwise, i.e., $f \in \partial\Phi(u)$ if and only if

$$u, f \in L^2(\Omega) \text{ and } f(x) \in \partial\varphi(u(x)) , \quad \text{a.e. } x \in \Omega .$$

(See Proposition II.8.1.)

EXAMPLE 2.C: L^2 -REALIZATIONS.

Let A be m -accretive on the Hilbert space H . Set $\mathcal{H} = L^2(\Omega, H)$, the square-summable H -valued functions on Ω , and define $\mathcal{A}$ on $\mathcal{H}$ by $v \in \mathcal{A}(u)$ if and only if

$$u, v \in \mathcal{H} \text{ and } v(x) \in A(u(x)) , \quad \text{a.e. } x \in \Omega .$$

If $A(0) \ni 0$ or if Ω has finite measure, it follows easily that $\mathcal{A}$ is m -accretive and $(I + \alpha\mathcal{A})^{-1}f(x) = (I + \alpha A)^{-1}(f(x))$, a.e. $x \in \Omega$. Specifically, given $f \in \mathcal{H}$ we define $u(x) = (I + \alpha A)^{-1}(f(x))$, $x \in \Omega$, and check that $u \in \mathcal{H}$.

Suppose additionally that A is a subgradient, i.e., $A = \partial\varphi$ for an appropriate $\varphi : H \rightarrow \mathbb{R}_\infty$. Define $\Phi : \mathcal{H} \rightarrow \mathbb{R}_\infty$ as above and note that $\mathcal{A} \subset \partial\Phi$. But $\mathcal{A}$ is maximal, so we see $\mathcal{A} = \partial\Phi$. Moreover, for $\alpha > 0$ we have $\mathcal{A}_\alpha u(x) = A_\alpha(u(x))$ and this leads to

$$\begin{aligned} \Phi_\alpha(u) &= \frac{\alpha}{2} \|\mathcal{A}_\alpha u\|^2 + \Phi((I + \alpha\mathcal{A})^{-1}u) \\ &= \int_\Omega \left\{ \frac{\alpha}{2} \|A_\alpha(u(x))\|^2 + \varphi(J_\alpha(u(x))) \right\} dx \\ &= \int_\Omega \varphi_\alpha(u(x)) dx . \end{aligned}$$

Thus the approximations of Φ and φ correspond.

EXAMPLE 2.D: INITIAL-VALUE PROBLEMS.

Let H be the Hilbert space $L^2(0, 1)$ and $u_0 \in \mathbb{R}$. Define $A = \frac{d}{dt}$ on the domain $D(A) = \{u \in H^1(0, 1) : u(0) = u_0\}$. Then A is m -accretive and the resolvent is given by

$$(2.1) \quad (I + \alpha A)^{-1}f(t) = e^{-t/\alpha}u_0 + \frac{1}{\alpha} \int_0^t e^{-\frac{(t-s)}{\alpha}} f(s) ds .$$

Note that this is of the form $e^{-t/\alpha}u_0 + (1 - e^{-t/\alpha})w(t)$, where $w(t)$ is a convex combination of the values of f and thus $(I + \alpha A)^{-1}f(t)$ is a convex combination of u_0 and $w(t)$. This observation is very useful when considering data in a given closed convex set, for then the resolvent remains in that set.

The above extends directly to the case of vector-valued functions, i.e., on $\mathcal{H} = L^2((0, 1), H)$. Moreover, it can be extended to $\mathcal{H} = L^2(\Omega, H)$ where Ω is a smoothly bounded domain in $\mathbb{R}^n$, $A = \frac{\partial}{\partial x_k}$, $1 \leq k \leq n$, and $D(A) = \{u \in \mathcal{H} : Au \in \mathcal{H} \text{ and } u|_{\Gamma_k} = u_0\}$ where $\Gamma_k = \{y \in \partial\Omega : \nu_k(y) < 0\}$. Here ν_k is the k -th component of the unit outward normal vector $\vec{\nu}$ on $\partial\Omega$.

EXAMPLE 2.E: BOUNDARY-VALUE PROBLEMS.

Let Ω be a bounded domain in $\mathbb{R}^n$ whose boundary $\Gamma = \partial\Omega$ is a C^1 -manifold of dimension $n - 1$ with Ω locally on one side. Thus we have a *trace* functional $\gamma : H^1(\Omega) \rightarrow L^2(\Gamma)$ which gives generalized boundary values. (See Section II.4.) Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be convex and continuous with

$$(2.2) \quad 0 \leq \psi(s) \leq C(s^2 + 1) , \quad s \in \mathbb{R} ,$$

and define $\Psi : H = L^2(\Omega) \rightarrow \mathbb{R}_\infty$ by

$$\Psi(u) \equiv \frac{1}{2} \int_\Omega |\vec{\nabla} u|^2 dx + \int_\Gamma \psi(\gamma u) ds \text{ if } u \in H^1(\Omega) , +\infty \text{ otherwise} .$$

Note that $\text{dom}(\Psi) = H^1(\Omega)$ and that Ψ is proper and convex. To see Ψ is lower-semi-continuous, let $u_n \rightarrow u$ in $L^2(\Omega)$ with $\Psi(u_n) \leq c$; after passing to a subsequence we have $u_n \rightharpoonup u$ in $H^1(\Omega)$ and $\gamma u_n \rightharpoonup \gamma u$ in $L^2(\Gamma)$, so we have

$$\frac{1}{2} \int_{\Omega} |\vec{\nabla} u|^2 dx \leq \liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\Omega} |\vec{\nabla} u_n|^2 dx, \quad \int_{\Gamma} \psi(\gamma u) ds \leq \liminf_{n \rightarrow \infty} \int_{\Gamma} \psi(\gamma u_n) ds$$

by weak-lower-semicontinuity of the respective terms. The result follows by the super-additivity of $\liminf$.

To compute the subgradient, let $F \in \partial\Psi(u)$, that is, $F \in L^2(\Omega)$, $u \in H^1(\Omega)$ and

$$\int_{\Omega} F(v - u) dx \leq \Psi(v) - \Psi(u), \quad v \in H^1(\Omega).$$

Let $w \in H^1(\Omega)$, $0 \leq t \leq 1$, and set $v = tw + (1 - t)u$ above, use convexity of ψ and then let $t \rightarrow 0^+$ to obtain

$$(2.3) \quad \int_{\Omega} F(w - u) dx \leq \int_{\Omega} \vec{\nabla} u \cdot \vec{\nabla} (w - u) dx + \int_{\Gamma} (\psi(\gamma w) - \psi(\gamma u)) dx, \quad w \in H^1(\Omega).$$

Choosing $w = u \pm \varphi$ with $\varphi \in C_0^\infty(\Omega)$ in (2.3) yields the partial differential equation

$$(2.4.a) \quad -\Delta u = F \quad \text{in } L^2(\Omega)$$

in the sense of distributions on Ω . Let $B = \gamma[H^1(\Omega)]$ be the range of γ with the norm induced by the quotient map from $H^1(\Omega)/H_0^1(\Omega)$. Then according to Proposition II.5.3 we have the *abstract Green's theorem*

$$\int_{\Omega} \vec{\nabla} u \cdot \vec{\nabla} v dx = \int_{\Omega} (-\Delta u)v dx + \partial u(\gamma v), \quad v \in H^1(\Omega)$$

with a $\partial u \in B'$ which extends the notion of normal derivative $\frac{\partial}{\partial \nu} = \vec{\nu} \cdot \vec{\nabla}$ on the boundary Γ , so we obtain

$$\partial u(\zeta - \gamma u) \leq \int_{\Gamma} (\psi(\zeta) - \psi(\gamma u)) ds, \quad \zeta \in B.$$

Using the growth-estimate (2.2) on ψ gives a $c_1 > 0$ such that

$$|\partial u(\zeta)| \leq c_1(\|\zeta\|_{L^2(\Gamma)}^2 + 1), \quad \zeta \in B,$$

and B is dense in $L^2(\Gamma)$, so we obtain $\partial u \in L^2(\Gamma)$ and the generalized *boundary condition*

$$(2.4.b) \quad \partial u + \partial\psi(\gamma u) \ni 0 \quad \text{in } L^2(\Gamma).$$

Thus $F \in \partial\Psi(u)$ implies (2.4), and the converse is immediate, so this characterizes $\partial\Psi$.

It is very useful to be able to *add* operators. The problem is that the domain of the sum is the intersection of the domains, and this can be too small unless the operators are compatible. For example, the two operators $-\Delta$ on $D_1 = H_0^1 \cap H^2$ (Dirichlet) and $-\Delta$ on $D_2 = \{u \in H^2 : \frac{\partial u}{\partial \nu} = 0\}$ (Neuman) are both m -accretive, but $D_1 \cap D_2$ is too small.

LEMMA 2.1. *If A is m -accretive and B is accretive and Lipschitz, then $A + B$ is m -accretive.*

PROOF. First note that $A + B$ is clearly accretive. Given $f \in H$ we want $u \in \text{dom}(A) : f \in u + Bu + A(u)$. Since A is m -accretive if and only if αA is m -accretive $\forall \alpha > 0$, we may assume B is a strict contraction. Thus, the desired u is characterized by $u = (I + A)^{-1}(f - Bu)$, the fixed point of a *strict contraction* on H . $\square$

Consider a pair A, B of m -accretive operators, and the equation for their sum, $f \in u + Au + Bu$. A “penalty method” of approximation is the system

$$u_\alpha + Au_\alpha + \frac{1}{\alpha}(u_\alpha - v_\alpha) \ni f, \quad Bv_\alpha + \frac{1}{\alpha}(v_\alpha - u_\alpha) \ni 0$$

in which $\frac{1}{\alpha}(u_\alpha - v_\alpha)$ is the “penalty”: as $\alpha \rightarrow 0$ one hopes $u_0 \cong v_0$ in the limit. This system is equivalent to the single equation

$$(2.5) \quad u_\alpha + A(u_\alpha) + B_\alpha(u_\alpha) \ni f$$

which has a solution by Lemma 2.1.

PROPOSITION 2.1 (BREZIS-CRANDALL-PAZY). *Let A and B be m -accretive. If the sequence $\{B_\alpha u_\alpha\}$ is bounded, then there exists $u : f \in u + Au + Bu$ and $u_\alpha \rightarrow u$.*

PROOF. Set $w_\alpha = f - u_\alpha - B_\alpha u_\alpha \in A(u_\alpha)$. Then

$$\|u_\alpha - u_\beta\|^2 + (w_\alpha - w_\beta, u_\alpha - u_\beta) + (B_\alpha u_\alpha - B_\beta u_\beta, u_\alpha - u_\beta) = 0$$

and $u_\alpha - u_\beta = (\alpha B_\alpha u_\alpha - \beta B_\beta u_\beta) + ((I + \alpha B)^{-1}u_\alpha - (I + \beta B)^{-1}u_\beta)$ imply that

$$\|u_\alpha - u_\beta\|^2 \leq \|B_\alpha u_\alpha - B_\beta u_\beta\| \|\alpha B_\alpha u_\alpha - \beta B_\beta u_\beta\|$$

so $\{u_\alpha\}$ is Cauchy and $u_\alpha \rightarrow u \in H$. Since $\{B_\alpha u_\alpha\}$ and $\{w_\alpha\}$ are bounded, for some subsequence we have $B_\alpha u_\alpha \rightarrow v$, $w_\alpha \rightarrow w$. But then $w \in A(u)$; since $(I + \alpha B)^{-1}u_\alpha = -\alpha B_\alpha u_\alpha + u_\alpha \rightarrow u$ and $B((I + \alpha B)^{-1}u_\alpha) \ni f - u_\alpha - w_\alpha$, we have $v \in B(u)$. Thus

$$u + w + v = f, \quad w \in A(u), v \in B(u). \quad \square$$

In order to use this result, one needs an *a-priori estimate* on $\{B_\alpha u_\alpha\}$. One can show also that $\{B_\alpha u_\alpha\}$ is bounded *only* if there exists a solution. Here is a useful observation.

LEMMA 2.2. *If $D(A) \cap D(B) \neq \varphi$, then $\{u_\alpha\}$ is bounded.*

PROOF. Let $u_0 \in D(A) \cap D(B)$ and pick $f_\alpha \in u_0 + A(u_0) + B_\alpha(u_0)$. Since A, B_α are accretive, $|u_\alpha - u_0|^2 \leq (f - f_\alpha, u_\alpha - u_0)$, hence $|u_\alpha - u_0| \leq |f - f_\alpha|$; note $|B_\alpha u_0| \leq |B^0 u_0|$. $\square$

Thus, if also A or B is bounded, Proposition 2.1 applies.

PROPOSITION 2.2 (BREZIS). *Let B be m -accretive and $\varphi : H \rightarrow \mathbb{R}_\infty$ be proper, convex and lower semi-continuous. Suppose there is a $C > 0$ such that*

$$(2.6) \quad \varphi((I + \alpha B)^{-1}u) \leq \varphi(u) + C\alpha, u \in H, \alpha > 0.$$

Then $\partial\varphi + B$ is m -accretive.

PROOF. Let $f \in H$ and u_α the solution for $\alpha > 0$ of (2.5),

$$u_\alpha + \partial\varphi(u_\alpha) + B_\alpha(u_\alpha) \ni f .$$

Thus we have

$$(2.7) \quad (f - B_\alpha(u_\alpha) - u_\alpha, v - u_\alpha) \leq \varphi(v) - \varphi(u_\alpha) , \quad v \in H .$$

Set $v = (I + \alpha B)^{-1}(u_\alpha)$ to get

$$(f - B_\alpha(u_\alpha) - u_\alpha, -\alpha B_\alpha(u_\alpha)) \leq C\alpha , \quad \alpha > 0 ,$$

from which there follows

$$\|B_\alpha(u_\alpha)\|^2 \leq \|f - u_\alpha\| \|B_\alpha(u_\alpha)\| + C .$$

If $D(\partial\varphi) \cap D(B) \neq \emptyset$, then by Lemma 2.2 we are done. In general, it suffices to set $v = v_0 \in \text{dom}(\varphi) \cap D(B)$ in (2.7) to get

$$(f - B_\alpha(u_\alpha) - u_\alpha, v_0 - u_\alpha) \leq \varphi(v_0) - \varphi(u_\alpha) .$$

Since B_α is accretive we have

$$(f - B_\alpha(v_0) - u_\alpha, v_0 - u_\alpha) \leq \varphi(v_0) - \varphi(u_\alpha) ;$$

since φ has an affine lower bound, this shows $\{u_\alpha\}$ is bounded, and we are done. $\square$

Note that if φ is the indicator function of the closed, convex, non-empty set K , $\varphi = I_K$ where $I_K(u) = \begin{cases} 0 , & u \in K \\ +\infty , & u \notin K \end{cases}$, then (2.6) is equivalent to $(I + \alpha B)^{-1}[K] \subset K$. For additional characterizations of (2.6), see Proposition 5.4.

EXAMPLE 2.F: DOUBLY-NONLINEAR BOUNDARY-VALUE-PROBLEM.

Here we add Examples 2.B and 2.E:

$$\Phi(u) \cong \int_{\Omega} \varphi(u(x)) dx , \quad \Psi(u) = \frac{1}{2} \int_{\Omega} |\vec{\nabla} u|^2 dx + \int_{\Gamma} \psi(\gamma u) ds$$

It will follow that $\partial\Phi + \partial\Psi$ is m -accretive and equal to $\partial(\Phi + \psi)$ if we can show

$$\Psi((I + \alpha\partial\Phi)^{-1}f) \leq \Psi(f) , \quad f \in H .$$

But this is immediate since $(I + \alpha\partial\varphi)^{-1}$ is a monotone contraction on $\mathbb{R}$. In particular, it follows that for every $F \in L^2(\Omega)$ there is a unique

$$(2.8) \quad \begin{aligned} u &\in H^1(\Omega) : -\Delta u + \partial\varphi(u) + u \ni F \text{ in } L^2(\Omega) \\ \partial u + \partial\psi(\gamma u) &\ni 0 \text{ in } L^2(\Gamma) \end{aligned}$$

when ψ is quadratically-bounded. (The general case is similar when the boundary condition is interpreted in B' .) This is a *doubly-nonlinear* elliptic boundary-value problem.

EXAMPLE 2.G: CAUCHY PROBLEM.

This time we add Examples 2.C and 2.D: we let H be a Hilbert space and set

$$\Phi(u) = \int_0^1 \varphi(u(t)) dt, \quad A = \frac{d}{dt} \text{ with } u(0) = u_0 \text{ on } \mathcal{H} = L^2(0, 1; H)$$

where the function $\varphi : H \rightarrow \mathbb{R}_\infty^+$ is convex and LSC as in Example 2.C. Recall from (2.1) that

$$(I + \alpha A)^{-1} f(t) = u_0 e^{-t/\alpha} + (1 - e^{-t/\alpha}) \frac{1}{\alpha(1 - e^{-t/\alpha})} \int_0^t e^{-(t-s)/\alpha} f(s) ds.$$

From here by convexity of φ we get

$$\varphi((I + \alpha A)^{-1} f(t)) \leq e^{-t/\alpha} \varphi(u_0) + (1 - e^{-t/\alpha}) \frac{1}{\alpha(1 - e^{-t/\alpha})} \int_0^t e^{-(t-s)/\alpha} \varphi(f(s)) ds,$$

and integrating this over $(0, 1)$ gives

$$(2.9) \quad \Phi((I + \alpha A)^{-1} f) \leq \alpha(1 - e^{-1/\alpha}) \varphi(u_0) + \Phi(f).$$

Thus (2.6) is satisfied with $C = \varphi(u_0)$ if $u_0 \in \text{dom}(\varphi)$.

An alternative derivation of (2.9) is as follows. Set $u_\alpha = (I + \alpha A)^{-1} f$ so that

$$\alpha \frac{du_\alpha}{dt} + u_\alpha = f, \quad u_\alpha(0) = u_0,$$

and let $\varphi_\beta, \beta > 0$, be given by Proposition 1.8. Then we have

$$\begin{aligned} \varphi_\beta(f(t)) - \varphi_\beta(u_\alpha(t)) &\geq \left(\varphi'_\beta(u_\alpha(t)), f(t) - u_\alpha(t) \right) \\ &= \left(\varphi'_\beta(u_\alpha(t)), \alpha \frac{du_\alpha}{dt} \right) \\ &= \alpha \frac{d}{dt} \varphi_\beta(u_\alpha(t)) \end{aligned}$$

and an integration gives (by Example 2.C above)

$$\Phi_\beta(u_\alpha) \leq \Phi_\beta(f) + \alpha \varphi_\beta(u_0).$$

Taking the limit as $\beta \rightarrow 0$ yields (2.6). Thus it follows that if $u_0 \in \text{dom}(\varphi)$ then $A + \partial\Phi$ is m -accretive on $L^2((0, 1), H)$. This corresponds to an *abstract Cauchy problem* in the Hilbert space H . That is, $F \in (A + \partial\Phi)(u)$ is characterized by

$$\begin{aligned} \frac{du(t)}{dt} + \partial\varphi(u(t)) &\ni F(t), \quad \text{a.e. } t \in (0, 1), \\ u(0) &= u_0. \end{aligned}$$

In particular, if we apply this to Example 2.F by choosing $\varphi = \Phi + \Psi$ on $H = L^2(0, 1)$ from Example 2.B and Example 2.E, with the obvious duplicated use of φ , it follows that $\alpha I + A + \partial\Phi + \partial\Psi$ is surjective for each $\alpha > 0$. Thus, for every

$$F \in L^2((0, 1); L^2(\Omega)) \cong L^2((0, 1) \times \Omega)$$

and $u_0 \in \text{dom}(\Phi + \Psi)$ there is a unique $u \in L^2((0, 1), H^1(\Omega))$ such that

$$(2.10.a) \quad \alpha u + \frac{\partial u}{\partial t} - \Delta u + \partial\varphi(u) \ni F \text{ in } L^2((0, 1) \times \Omega) ,$$

$$(2.10.b) \quad \partial u + \partial\psi(\gamma u) \ni 0 \text{ in } L^2((0, 1) \times \Gamma) ,$$

$$(2.10.c) \quad u(0, \cdot) = u_0 \text{ in } L^2(\Omega) .$$

This is a *doubly-nonlinear* parabolic initial-boundary-value problem.

REMARK. If $\partial\varphi$ is *strongly-accretive*, i.e., if $\partial\varphi - \alpha I$ is accretive for some $\alpha > 0$, then the preceding results hold also for $\alpha = 0$ by a change of notation. More generally, consider the Cauchy problem in the Hilbert space H

$$\frac{du}{dt} + \alpha u + A(u(t)) \ni f(t) , \quad 0 < t < 1 , \quad u(0) = u_0$$

with an m -accretive A in H . Let $v(t) = e^{\alpha t}u(t)$ to get an equivalent problem

$$\frac{dv}{dt} + e^{\alpha t}A(e^{-\alpha t}v(t)) \ni e^{\alpha t}f(t) , \quad 0 < t < 1 , \quad v(0) = u_0 .$$

Take scalar product with $w \in L^2((0, 1), H)$ to get at a.e. $t \in (0, 1)$

$$\left(\frac{dv(t)}{dt}, w(t) \right)_H + (A(e^{-\alpha t}v(t)), e^{\alpha t}w(t))_H = (e^{\alpha t}f(t), w(t))_H .$$

But in order to exploit the accretive estimates on A we need to multiply by $e^{-2\alpha t}$ before integrating, and then we have

$$\begin{aligned} & \int_0^1 (v'(t), w(t))_H e^{-2\alpha t} dt + \int_0^1 (A(e^{-\alpha t}v(t)), e^{-\alpha t}w(t)) dt \\ &= \int_0^1 (e^{\alpha t}f(t), w(t)) e^{-2\alpha t} dt . \end{aligned}$$

Thus in the space $\mathcal{H} = L^2((0, 1), H; e^{-2\alpha t} dt)$ with the indicated *density* or *weight* $e^{-2\alpha t}$ the operator $\mathcal{A}(v)(t) = e^{\alpha t}A(e^{-\alpha t}v(t))$ is accretive. The results of Example 3 carry over here and thereby one can replace A by $A + \alpha I$ for *any* $\alpha \in \mathbb{R}$.

It is not difficult to give sufficient additional conditions on an m -accretive operator A which imply that it is surjective. For example, if A is *strongly-accretive*, i.e., there is a $c > 0$ for which

$$(y_1 - y_2, x_1 - x_2) \geq c\|x_1 - x_2\|^2 , \quad [x_j, y_j] \in A , \quad j = 1, 2 ,$$

then $A - cI$ is m -accretive, hence, $A = (A - cI) + cI$ is surjective. For another example, consider $A = \partial\varphi$. If $f \in H$ and

$$\lim_{\|v\| \rightarrow \infty} (\varphi(v) - (f, v)) = \infty ,$$

then the convex function $\varphi(\cdot) - (f, \cdot)$ attains a minimum at some $u \in H$,

$$\varphi(u) - (f, u) \leq \varphi(v) - (f, v) , \quad v \in H ,$$

hence, $f \in \partial\varphi(u) = A(u)$. This holds for every $f \in H$ if

$$\lim_{\|u\| \rightarrow \infty} \frac{\varphi(u)}{\|u\|} = \infty .$$

This is a *coercivity* condition on φ , a convenient sufficient condition for $\partial\varphi$ to be surjective. More generally we have the following.

PROPOSITION 2.3. *Let A be m -accretive and let A^{-1} be bounded, that is,*

$$\lim_{\|v\| \rightarrow \infty} \|A^0(v)\| = +\infty .$$

Then A is surjective: $Rg(A) = H$.

PROOF. Note that $Rg(A)$ is closed, for if $y_n \rightarrow y$ in H with $y_n \in Rg(A)$, then there is a sequence $x_n \in A^{-1}(y_n)$ which by assumption is bounded. By passing to a subsequence, which we denote again by $\{x_n\}$, we have $x_n \rightharpoonup x$ and $[x, y] \in A$ by Proposition 1.6.

Let $[x_0, y_0] \in A$ and $y \in H$. For each $\alpha > 0$ set $x_\alpha \equiv (\alpha I + A)^{-1}(y + \alpha x_0)$ and $y_\alpha \equiv y + \alpha(x_0 - x_\alpha) \in A(x_\alpha)$. Since A is accretive we obtain $(y_\alpha - y_0, x_\alpha - x_0) \geq 0$ and this shows $(y_\alpha - y_0, y - y_\alpha) \geq 0$. Consequently follow $\|y_\alpha - y_0\|^2 \leq (y_\alpha - y_0, y - y_0)$ and $\|y_\alpha - y_0\| \leq \|y - y_0\|$. Thus $\{y_\alpha\}$ is bounded and by assumption $\{x_\alpha\}$ is bounded. This shows $y_\alpha \rightarrow y \in \overline{Rg(A)} = Rg(A)$. $\square$

Proposition 2.3 asserts that for an m -accretive operator A to be surjective, it suffices to know *a-priori estimates* on solutions, i.e., if $A(u) \ni f$, then $\|u\| \leq C(\|f\|)$ where $C : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a *bounded* function. Such estimates often occur as a *coercive* property of A :

$$\lim_{\|u\| \rightarrow \infty} \frac{(A^0 u, u)}{\|u\|} = \infty .$$

Note that it is not true that A^{-1} being bounded implies that A is coercive. For example, the operator $A[x_1, x_2] = [-x_2, x_1]$ on $\mathbb{R}^2$ is not coercive.

It is not the case that coercivity is preserved by addition. For example, set $A = \frac{d}{dt}$ on $D(A) = \{u \in H^1(0, 1) : u(0) = 0\}$ and $B = -\frac{d}{dt}$ on $D(B) = \{u \in H^1(0, 1) : u(1) = 0\}$. They are both m -accretive and coercive on $L^2(0, 1)$, but $A + B = 0$. From the calculation $\|(A + B)u\|^2 = \|Au\|^2 + \|Bu\|^2 + 2(Au, Bu)_H$, it is clear that one needs some control of the “angle” between Au and Bu . This does occur in the situation of Proposition 2.2.

COROLLARY 2.1. *In the situation of Proposition 2.2, we obtain the estimate*

$$\|B^0 u\| \leq \|(\partial\varphi + B)^0 u\| + \sqrt{C}$$

Thus $(\partial\varphi + B)^{-1}$ is bounded if B^{-1} or $(\partial\varphi)^{-1}$ is bounded.

PROOF. For $w \in \partial\varphi(u)$ we have

$$\varphi((I + \alpha B)^{-1}u) - \varphi(u) \geq (w, (I + \alpha B)^{-1}u - u)$$

and by (2.6) we have $\alpha C \geq (w, -\alpha B_\alpha u)$, $\alpha > 0$. Thus for $u \in D(B) \cap D(\partial\varphi)$ there follows

$$(B^0(u), w) \geq -C , \quad w \in \partial\varphi(u) .$$

Let $f \in (\partial\varphi + B)(u)$, that is,

$$f = v + w , \text{ where } v \in B(u) , w \in \partial\varphi(u) .$$

Then

$$(B^0(u), f) = (B^0(u), v) + (B^0(u), w) \geq \|B^0(u)\|^2 - C ,$$

and the desired estimate follows from this. $\square$

EXAMPLE 2.G (CONTINUED). Since A is coercive it follows that $A + \partial\Phi + \partial\Psi$ is coercive on $\mathcal{H} = L^2(0, 1; H)$ and, hence, (again) we find for every $F \in L^2((0, 1), L^2(\Omega))$ and $u_0 \in \text{dom}(\Phi + \Psi)$ there is a unique solution of (2.10) with $\alpha = 0$.

We close with some equivalent notions of coercivity for subgradients.

PROPOSITION 2.4 (BREZIS). *Let $\varphi : H \rightarrow \mathbb{R}_\infty^+$ be proper, convex and lower semi-contin-u-ous, and set $A = \partial\varphi$. The following are equivalent:*

- (a) $\lim_{\|u\| \rightarrow \infty} \left(\frac{\varphi(u)}{\|u\|} \right) = \infty$.
- (b) for every $u_0 \in \text{dom}(\varphi)$, $\lim_{\substack{\|u\| \rightarrow \infty \\ [u, f] \in A}} \frac{(f, u - u_0)}{\|u\|} = \infty$.
- (c) there exists $u_0 \in H$ such that $\lim_{\substack{\|u\| \rightarrow \infty \\ u \in \text{dom}(A)}} \frac{(A^0 u, u - u_0)}{\|u\|} = \infty$.
- (d) $\lim_{\substack{\|u\| \rightarrow \infty \\ u \in \text{dom}(A)}} \|A^0 u\| = \infty$.
- (e) A^{-1} is bounded, hence, $Rg(A) = H$.

PROOF. From the inequality $\varphi(u) \leq \varphi(u_0) + (f, u - u_0)$ it follows that (a) $\Rightarrow$ (b). Clearly (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e). To show (e) $\Rightarrow$ (a), let $R > 0$ and note that by (e), for every $z \in H$ with $\|z\| \leq R$ there is a $v : A(v) \ni z$ and $\|v\| \leq M$. Thus from

$$\varphi(u) - \varphi(v) \geq (z, u - v), \quad u \in \text{dom}(\varphi),$$

we obtain

$$(z, u) \leq \varphi(u) + MR \quad \forall \quad z \in H, \|z\| \leq R.$$

Hence $R\|u\| \leq \varphi(u) + MR$ and

$$\frac{\varphi(u)}{\|u\|} \geq R - \frac{MR}{\|u\|},$$

and so we obtain

$$\liminf_{\|u\| \rightarrow \infty} \frac{\varphi(u)}{\|u\|} \geq R$$

for every $R > 0$. □

IV.3. The Cauchy Problem in Hilbert space

Let A be a m -accretive operator in the Hilbert space H . As before we denote by $J_\alpha = (I + \alpha A)^{-1}$ the resolvent, by $A_\alpha = \frac{1}{\alpha}(I - J_\alpha)$ the Yosida approximation, and by A^0 the minimal section of A . We shall study the evolution equation

$$(3.1) \quad \frac{du}{dt}(t) + A(u(t)) \ni 0, \quad 0 < t.$$

A *solution* of (3.1) is a continuous function $u : [0, \infty) \rightarrow H$ which is absolutely continuous on each $[a, b]$, $0 < a < b$, hence, u is differentiable a.e. with $\frac{du}{dt} \in L^1(a, b; H)$, and for a.e. $t > 0$, $u(t) \in D(A)$ and (3.1) holds. The *Cauchy problem* is to find a solution of (3.1) with $u(0) = u_0$ given in H . The uniqueness of a solution of the Cauchy problem is immediate when A is accretive or Lipschitz. Existence

is similarly easy when A is Lipschitz; existence of a solution for the m -accretive case is the goal of this section. This will be done by replacing A by the Lipschitz A_α , $\alpha > 0$, to obtain a sequence of solutions u_α of this *approximate* or *regularized* problem and then showing $\lim_{\alpha \rightarrow 0} u_\alpha$ exists and is a solution of the Cauchy problem for (3.1).

The Yosida approximation of A in (3.1) and a re-scaled time variable suggest that we consider the differential equation

$$(3.2) \quad \frac{du}{dt}(t) + u(t) - J(u(t)) = 0$$

in which J is Lipschitz.

LEMMA 3.1. *Assume K is a closed, convex non-empty set in H and $J : K \rightarrow K$ is Lipschitz: $\|J(x) - J(y)\| \leq L\|x - y\|$, $x, y \in K$. Then for each $u_0 \in K$ there exists a unique absolutely continuous $u : [0, \infty) \rightarrow H$ which is a solution of (3.2) with $u(t) \in K$, $t > 0$, and with $u(0) = u_0$.*

REMARK. We shall need below only the case $K = H$ in Lemma 3.1. However, that the equation (3.2) would leave the set K invariant is suggested by the observation that for each u on the boundary of K the vector $J(u) - u$ is directed into K , hence, the equation (3.2) implies that the direction of the solution is back into K .

PROOF. If u is such a solution then

$$\frac{d}{dt}(e^t u(t)) = e^t J(u(t)) , \quad t > 0 ,$$

so we find by integrating that

$$(3.3) \quad u(t) = e^{-t} u_0 + \int_0^t e^{s-t} J(u(s)) ds , \quad t \geq 0 .$$

Let's solve (3.3) in the Banach space $C([0, T], H)$ by a fixed-point theorem. Consider the closed, convex $\mathcal{K} = \{u \in C([0, T], H) : u(t) \in K \text{ for all } t \in [0, T]\}$, and define

$$\mathbb{T}(u)(t) = e^{-t} u_0 + \int_0^t e^{s-t} J u(s) ds , \quad 0 \leq t \leq T ,$$

for $u \in \mathcal{K}$. Then $\mathbb{T}(u) \in C([0, T], H)$ and the indicated convex combination satisfies

$$v_1(t) = \int_0^t e^{s-t} J(u(s)) ds / \int_0^t e^{s-t} ds \in K , \quad t > 0 .$$

Note that $\int_0^t e^{s-t} ds = 1 - e^{-t}$ so

$$\mathbb{T}(u)(t) = e^{-t} u_0 + (1 - e^{-t}) v_1(t) \in K , \quad t > 0 .$$

Thus $\mathbb{T}$ maps $\mathcal{K}$ into $\mathcal{K}$. For any pair $u_1, u_2 \in \mathcal{K}$ we have

$$\|\mathbb{T}(u_1)(t) - \mathbb{T}(u_2)(t)\| \leq L \int_0^t e^{s-t} \|u_1(s) - u_2(s)\| ds$$

and this shows, successively,

$$\begin{aligned} \|\mathbb{T}u_1 - \mathbb{T}u_2\|_{C([0,t],H)} &\leq Lt\|u_1 - u_2\|_{C([0,T],H)} , \\ \|\mathbb{T}^2(u_1(t)) - \mathbb{T}^2(u_2(t))\| &\leq L \int_0^t e^{s-t} Ls \, ds \|u_1 - u_2\|_{C([0,T],H)} \\ &\leq \frac{L^2 t^2}{2} \|u_1 - u_2\|_{C([0,T],H)} \end{aligned}$$

and by induction we obtain for $k \geq 1$

$$\|\mathbb{T}^k u_1 - \mathbb{T}^k u_2\|_{C([0,t],H)} \leq \frac{(Lt)^k}{k!} \|u_1 - u_2\|_{C([0,T],H)} , \quad 0 \leq t \leq T .$$

Thus, $\mathbb{T}^k$ is a strict-contraction for k sufficiently large, so (3.3) has a unique solution in $\mathcal{K}$. $\square$

COROLLARY 3.1. *If $u_0 \in K$ and $\alpha > 0$ there exists a unique solution (as above) of*

$$(3.4) \quad \frac{du}{dt} + \frac{1}{\alpha} (u(t) - J(u(t))) = 0$$

with $u(t) \in \mathcal{K}$, and it is characterized by

$$u(t) = e^{-t/\alpha} u_0 + \frac{1}{\alpha} \int_0^t e^{(s-t)/\alpha} J(u(s)) \, ds .$$

PROOF. The change-of-variable $v(t) = u(\alpha t)$ shows (3.4) is equivalent to (3.2). $\square$

We shall show the Cauchy problem for (3.1) has a solution if $u(0) = u_0 \in D(A)$. For each $\alpha > 0$ let u_α be the solution of

$$(3.5) \quad \frac{du_\alpha}{dt}(t) + A_\alpha(u_\alpha(t)) = 0 , \quad t \geq 0 ,$$

with $u_\alpha(0) = u_0$ as is given in $C^1([0, \infty), H)$ by Corollary 3.1. If $h > 0$ then $u_\alpha(t+h)$ is a solution of (3.5) and since A_α is accretive we obtain

$$\frac{d}{dt} \|u_\alpha(t+h) - u_\alpha(t)\|^2 = -2 (A_\alpha(u_\alpha(t+h)) - A_\alpha(u_\alpha(t)), u_\alpha(t+h) - u_\alpha(t))_H \leq 0$$

so $\|u_\alpha(t+h) - u_\alpha(t)\| \leq \|u_\alpha(h) - u_\alpha(0)\|$. Letting $h \rightarrow 0$ shows $\|u'_\alpha(t)\| \leq \|u'_\alpha(0)\|$, hence

$$(3.6) \quad \|A_\alpha(u_\alpha(t))\| \leq \|A_\alpha(u_0)\| \leq \|A^0 u_0\| , \quad t \geq 0 , \quad \alpha > 0 .$$

We shall show u_α is Cauchy in $C([0, T], H)$. For $\alpha, \beta > 0$

$$\frac{d}{dt} \|u_\alpha(t) - u_\beta(t)\|^2 = -2 (A_\alpha u_\alpha(t) - A_\beta u_\beta(t), u_\alpha(t) - u_\beta(t))_H .$$

Write $u_\alpha = \alpha A_\alpha u_\alpha + J_\alpha u_\alpha$ and likewise for u_β to get

$$\begin{aligned} (A_\alpha u_\alpha - A_\beta u_\beta, u_\alpha - u_\beta)_H &= (A_\alpha u_\alpha - A_\beta u_\beta, \alpha A_\alpha u_\alpha - \beta A_\beta u_\beta)_H \\ &\quad + (A_\alpha u_\alpha - A_\beta u_\beta, J_\alpha u_\alpha - J_\beta u_\beta)_H . \end{aligned}$$

The latter term is non-negative, since A is accretive, so the right side is bounded below by

$$\begin{aligned} & \alpha \|A_\alpha u_\alpha\|^2 + \beta \|A_\beta u_\beta\|^2 - (\alpha + \beta)(A_\alpha u_\alpha, A_\beta u_\beta)_H \\ & \geq \alpha \|A_\alpha u_\alpha\|^2 + \beta \|A_\beta u_\beta\|^2 - \alpha \left(\|A_\alpha u_\alpha\|^2 + \frac{1}{4} \|A_\beta u_\beta\|^2 \right) \\ & \quad - \beta \left(\|A_\beta u_\beta\|^2 + \frac{1}{4} \|A_\alpha u_\alpha\|^2 \right) \\ & \geq -\frac{\alpha + \beta}{4} \|A^0 u_0\|^2 . \end{aligned}$$

This shows

$$\frac{d}{dt} \|u_\alpha(t) - u_\beta(t)\|^2 \leq \frac{\alpha + \beta}{2} \|A^0 u_0\|^2$$

and, hence,

$$\|u_\alpha(t) - u_\beta(t)\| \leq \|A^0 u_0\| \left(\frac{\alpha + \beta}{2} t \right)^{1/2}, \quad t \geq 0 .$$

On each interval, $[0, T]$, the sequence $\{u_\alpha\}$ is uniformly Cauchy, hence, uniformly convergent to some $u \in C([0, T], H)$ with

$$\|u_\alpha(t) - u(t)\| \leq (\alpha T/2)^{1/2} \|A^0 u_0\|, \quad 0 \leq t \leq T, \quad \alpha > 0 .$$

From the estimate

$$\|J_\alpha u_\alpha(t) - u_\alpha(t)\| = \alpha \|A_\alpha(u_\alpha(t))\| \leq \alpha \|A^0 u_0\|, \quad \alpha > 0$$

it follows that $J_\alpha u_\alpha$ converges in $C([0, T], H)$ to u . The estimate (3.6) implies $\{u'_\alpha\}$ is bounded in the Hilbert space $L^2(0, T; H)$ so there is a subsequence which converges weakly. We take limits in

$$u_\alpha(t) = u_0 + \int_0^t u'_\alpha(s) ds$$

to find the limit is u' , and by uniqueness of weak limits the original sequence $\{u'_\alpha\}$ converges weakly to u' . Note that the realization $\mathcal{A}$ of A in $L^2(0, T; H)$ is m -accretive, i.e., $\mathcal{A}(u)(t) = A(u(t))$. Since $-u'_\alpha \in \mathcal{A}(J_\alpha u_\alpha)$ it follows from Proposition 1.6 that $-u' \in \mathcal{A}(u)$. This proves the first part of the following.

PROPOSITION 3.1 (KŌMURA-KATO). *Let A be m -accretive in the Hilbert space H . For each $u_0 \in D(A)$ there is a unique absolutely continuous $u : [0, \infty) \rightarrow H$ such that $u(0) = u_0$ and (3.1) holds at a.e. $t > 0$. Also, u is Lipschitz with $\|u'\|_{L^\infty(0, \infty; H)} \leq \|A^0 u_0\|$; at every $t \geq 0$, $u(t) \in D(A)$ and the right-derivative satisfies*

$$(3.7) \quad D^+ u(t) + A^0(u(t)) = 0 ;$$

and the function $t \mapsto A^0(u(t))$ is right-continuous with $\|A^0(u(t))\|$ decreasing.

PROOF (CONTINUED). Let $t \geq 0$. Since $J_\alpha(u_\alpha(t)) \rightarrow u(t)$ and (3.5) holds, it follows from Proposition 1.6 that $u(t) \in D(A)$. Moreover, some subsequence $A_{\alpha_n}(u_{\alpha_n}(t))$ converges weakly to a $w \in A(u(t))$ and by weak lower-semi-continuity of the norm

$$\|w\| \leq \liminf_{n \rightarrow \infty} \|A_{\alpha_n}(u_{\alpha_n}(t))\| \leq \|A^0(u_0)\|.$$

Thus, we obtain

$$\|A^0(u(t))\| \leq \|A^0(u_0)\|, \quad t \geq 0.$$

Let $t_n \downarrow 0$ with $A^0(u(t_n)) \rightarrow \xi$. Then Proposition 1.6 shows $\xi \in A(u_0)$ and $\|\xi\| \leq \liminf_{n \rightarrow \infty} \|A^0(u(t_n))\| \leq \|A^0(u_0)\|$, so $\xi = A^0(u_0)$ and we have shown $w = \lim_{n \rightarrow \infty} A^0(u(t_n)) = A^0(u_0)$. Furthermore,

$$\|A^0(u_0)\| \leq \liminf \|A^0(u(t_n))\| \leq \limsup \|A^0(u(t_n))\| \leq \|A^0(u_0)\|$$

so $\lim_{n \rightarrow \infty} \|A^0(u(t_n))\| = \|A^0(u_0)\|$. This shows the strong limit exists and satisfies $\lim_{n \rightarrow \infty} A^0(u(t_n)) = A^0(u_0)$, so $A^0 u(t)$ is continuous at $t = 0$. But for each $t_0 \geq 0$, the translate $u(t + t_0)$ is the solution of (3.1) with initial data $u(t_0)$, so $\lim_{n \rightarrow \infty} A^0(u(t_n + t_0)) = A^0(u(t_0))$. That is, $A^0(u(t))$ is right-continuous at every $t_0 \geq 0$; also $\|A^0(u(t + t_0))\| \leq \|A^0(u(t_0))\|$, so $\|A^0(u(t))\|$ is decreasing.

For all $t \geq 0$ and $h > 0$ we have similarly

$$\|u(t + h) - u(t)\| \leq h \|A^0(u(t))\|$$

from (3.6), so dividing by $h > 0$ and taking the limit show

$$\|u'(t)\| \leq \|A^0(u(t))\|, \quad \text{a.e. } t \geq 0.$$

With (3.1) this shows that

$$u'(t) + A^0(u(t)) = 0, \quad \text{a.e. } t \geq 0.$$

Finally, from the representation

$$u(t) = u_0 - \int_0^t A^0(u(s)) ds, \quad t \geq 0$$

and the right-continuity of the integrand we obtain (3.7) at every $t \geq 0$. $\square$

We shall next show that the Cauchy problem for (3.1) can be solved with initial data u_0 in $\overline{D(A)}$ for those operators A which are subgradients. Such operators are quite special, even within the linear examples, because the solution of (3.1) will satisfy $u(t) \in D(A)$ for $t > 0$ even though one starts with $u(0) \in \overline{D(A)}$. In examples with partial differential equations this means $u(t)$ is strictly smoother in its spatial variables than is $u(0)$, so this is generally called the *regularizing property* of (3.1).

In order to discuss this more easily we define for each $t \geq 0$ the function $S(t) : D(A) \rightarrow D(A)$ by $S(t)u_0 = u(t)$ where u is the solution of the Cauchy problem for (3.1) with $u(0) = u_0$ in the situation of Proposition 1. By taking the (uniformly) continuous extension of each $S(t)$ we obtain a family of functions $S(t) : \overline{D(A)} \rightarrow \overline{D(A)}$ which satisfies

$$(3.8.a) \quad \|S(t)u_1 - S(t)u_2\| \leq \|u_1 - u_2\|, \quad t \geq 0, \quad u_1, u_2 \in \overline{D(A)},$$

$$(3.8.b) \quad S(t_1 + t_2) = S(t_1) \circ S(t_2), \quad t_1, t_2 \geq 0,$$

$$(3.8.c) \quad S(\cdot)u_0 \text{ is continuous on } \mathbb{R}_+ \text{ for each } u_0 \in \overline{D(A)}.$$

This gives the (strongly) continuous *semigroup of contractions* generated by $-A$. The *regularizing property* is that $S(t) : \overline{D(A)} \rightarrow D(A)$ for $t > 0$.

PROPOSITION 3.2 (BREZIS). *Let $\varphi : H \rightarrow \mathbb{R}_\infty$ be proper, convex and lower-semi-continuous; set $A = \partial\varphi$ and let $\{S(t) : t \geq 0\}$ be the continuous semigroup of contractions generated by $-A$. Then for each $u_0 \in \overline{D(A)}$ and $t > 0$ it follows that $S(t)u_0 \in D(A)$ and*

$$(3.9) \quad \|A^0(S(t)u_0)\| \leq \|A^0v\| + \frac{1}{t}\|u_0 - v\|, \quad v \in D(A).$$

PROOF. From Proposition 1.8 we have $A_\alpha = \varphi'_\alpha$. Let u_α denote as before the solution of (3.5) with $u_\alpha(0) = u_0 \in \overline{D(A)}$. Fix $v \in H$ and $\alpha > 0$; define

$$\psi(w) = \varphi_\alpha(w) - \varphi_\alpha(v) - (A_\alpha v, w - v)_H, \quad w \in H$$

and note $\psi'(w) = \varphi'_\alpha(w) - A_\alpha v$, $w \in H$, $\min \psi = \psi(v) = 0$, and

$$u'_\alpha(t) + \psi'(u_\alpha(t)) = -A_\alpha v, \quad t \geq 0.$$

Since $(\psi'(u_\alpha), v - u_\alpha) \leq \psi(v) - \psi(u_\alpha) = -\psi(u_\alpha)$ it follows that

$$\begin{aligned} \psi(u_\alpha(t)) &\leq (u'_\alpha(t) + A_\alpha v, v - u_\alpha(t))_H \\ &= -\frac{1}{2} \frac{d}{dt} \|v - u_\alpha(t)\|^2 + (A_\alpha v, v - u_\alpha(t))_H, \end{aligned}$$

so we obtain by an integration

$$(3.10) \quad \int_0^T \psi(u_\alpha(t)) dt \leq \frac{1}{2} (\|v - u_0\|^2 - \|v - u_\alpha(T)\|^2) + \int_0^T (A_\alpha v, v - u_\alpha(t))_H dt.$$

This is the *energy estimate*. In order to estimate the derivative, u'_α , we take its scalar-product with its equation above, multiply by $t \geq 0$ and integrate to obtain, successively,

$$\begin{aligned} t\|u'_\alpha(t)\|^2 + t \frac{d}{dt} \psi(u_\alpha(t)) &= t \frac{d}{dt} (A_\alpha v, v - u_\alpha(t))_H, \\ \int_0^T t\|u'_\alpha(t)\|^2 dt + T\psi(u_\alpha(T)) - \int_0^T \psi(u_\alpha(t)) dt \\ &= T(A_\alpha v, v - u_\alpha(T))_H + \int_0^T (A_\alpha v, u_\alpha(t) - v)_H dt \end{aligned}$$

We combine this with (3.10) to get

$$\begin{aligned} \int_0^T t\|u'_\alpha(t)\|^2 dt &\leq \frac{1}{2} (\|v - u_0\|^2 - \|v - u_\alpha(T)\|^2) + T(A_\alpha v, v - u_\alpha(T))_H \\ (3.11) \quad &\leq \frac{1}{2} \|u_0 - v\|^2 + \frac{1}{2} T^2 \|A_\alpha v\|^2. \end{aligned}$$

The function $\|u'_\alpha(t)\| = \|A_\alpha^0(u_\alpha(t))\|$ is non-increasing in t , so the left side of (3.11) dominates $(T^2/2)\|u'_\alpha(T)\|^2$ and we obtain

$$\|A_\alpha u_\alpha(T)\|^2 \leq \frac{1}{T^2} \|u_0 - v\|^2 + \|A_\alpha v\|^2.$$

That is, we have the estimate

$$(3.12) \quad \|A_\alpha(u_\alpha(T))\| \leq \|A^0 v\| + \frac{1}{T} \|u_0 - v\|, \quad \alpha > 0.$$

Let's show $u_\alpha(T) \rightarrow S(T)u_0$ as $\alpha \rightarrow 0$. If $v_0 \in D(A)$ and v_α is the solution of (3.5) with $v(0) = v_0$, then

$$\begin{aligned} \|u_\alpha(T) - S(T)u_0\| &\leq \|u_\alpha(T) - v_\alpha(T)\| + \|v_\alpha(T) - S(T)v_0\| + \|S(T)v_0 - S(T)u_0\| \\ &\leq 2\|v_0 - u_0\| + \|v_\alpha(T) - S(T)v_0\|. \end{aligned}$$

Given $\varepsilon > 0$, choose v_0 and then α_0 so that $0 < \alpha \leq \alpha_0$ implies $\|u_\alpha(T) - S(T)u_0\| < \varepsilon$. From the estimate (3.12) and

$$\|J_\alpha(u_\alpha(T)) - u_\alpha(T)\| = \alpha \|A_\alpha(u_\alpha(T))\|$$

it is clear that $J_\alpha(u_\alpha(T)) \rightarrow S(T)u_0$. From Proposition 1.6 we obtain (as in the proof of Proposition 3.1) $S(T)u_0 \in D(A)$ and some subsequence $A_{\alpha_n}(u_{\alpha_n}(t)) \rightarrow y \in A(u(T))$, so (3.9) follows from (3.12) by weak lower-semi-continuity of the norm. $\square$

COROLLARY 3.2. *For each $u_0 \in \overline{D(A)}$ there is a unique solution u of (3.1) with $u(0) = u_0$. This solution satisfies the following:*

$u(t) \in D(A)$ for every $t > 0$;

u is Lipschitz on $[a, \infty)$ for every $a > 0$ with

$$\left\| \frac{du}{dt} \right\|_{L^\infty([a, \infty), H)} \leq \|A^0 v\| + \frac{1}{a} \|u_0 - v\|, \quad v \in D(A);$$

and the function $\varphi(u(t))$ is non-increasing and convex with right-derivative given at every $t > 0$ by

$$(3.13) \quad D^+ \varphi(u(t)) = -\|D^+ u(t)\|^2.$$

PROOF. It remains yet to verify (3.13). From this it will follow that $\varphi(u(\cdot))$ is non-increasing; since $\|D^+ u(t)\| = \|A^0(u(t))\|$ is non-increasing by Proposition 3.1, $\varphi(u(\cdot))$ is also convex. Furthermore, the non-increase of $\|D^+ u\|$ gives

$$\|u(t+h) - u(t)\| \leq \int_t^{t+h} \|D^+ u(s)\| ds \leq h \|D^+ u(a)\|, \quad t \geq a, \quad h > 0.$$

From (3.7), where $A^0 \subset \partial\varphi$, we obtain

$$\begin{aligned} -(D^+ u(t+h), u(t+h) - u(t))_H &\geq \varphi(u(t+h)) - \varphi(u(t)) \\ &\geq -(D^+ u(t), u(t+h) - u(t))_H \end{aligned}$$

and this gives with the preceding estimate

$$|\varphi(u(t+h)) - \varphi(u(t))| \leq h \|D^+ u(a)\|^2, \quad t \geq a, \quad h > 0,$$

so $\varphi(u(\cdot))$ is Lipschitz on $[a, \infty)$ with right-derivative given by (3.13). $\square$

The behavior of $u(t)$ and $\varphi(u(t))$ for t near zero is described by the following.

COROLLARY 3.3. *In the situation of Proposition 2, if $u_0 \in \overline{D(A)}$ then $\varphi(u) \in L^1(0, a)$ and $\sqrt{t} \frac{du(t)}{dt} \in L^2(0, a; H)$. Furthermore we have $u_0 \in \text{dom}(\varphi)$ if and only if $\frac{du}{dt} \in L^2(0, a; H)$ for each $a > 0$, and in that case*

$$(3.14) \quad \varphi(u_0) - \varphi(u(t)) = \int_0^t \left\| \frac{du}{dt} \right\|^2 ds, \quad t > 0,$$

so $\lim_{t \rightarrow 0} \varphi(u(t)) = \varphi(u_0)$ with monotone convergence, and

$$\begin{aligned} \|u(t) - u_0\| &\leq \sqrt{t} \left(\varphi(u_0) - \varphi(u(t)) \right)^{1/2}, \quad t > 0, \\ \sqrt{t} \|D^+ u(t)\| &\leq \left(\varphi(u_0) - \varphi(u(t)) \right)^{1/2}. \end{aligned}$$

PROOF. Since $J_\alpha(u_\alpha(t)) \rightarrow u(t)$, $t > 0$, and

$$\varphi(J_\alpha(u)) \leq \varphi_\alpha(u) = \frac{1}{2\alpha} \|u - J_\alpha u\|^2 + \varphi(J_\alpha u)$$

we deduce from the energy estimate (3.10) with $T = a$ by Fatou's Lemma that $\varphi(u) \in L^1(0, a)$. From the estimate (3.11) we obtain $\sqrt{t} \frac{du}{dt} \in L^2(0, a; H)$ from weak convergence of the derivative.

Let $u_0 \in \text{dom}(\varphi)$. From the counterpart of (3.13) for the approximate equation (3.5) follows

$$\varphi_\alpha(u_0) - \varphi_\alpha(u_\alpha(a)) = \int_0^a \|u'_\alpha(s)\|^2 ds$$

and, therefore, as above

$$\varphi(J_\alpha u_\alpha(a)) + \int_0^a \|u'_\alpha(s)\|^2 ds \leq \varphi_\alpha(u_0) \leq \varphi(u_0).$$

Since $J_\alpha u_\alpha(a) \rightarrow u(a)$ as $\alpha \rightarrow 0$ we find in the limit

$$\varphi(u(a)) + \int_0^a \|u'(s)\|^2 ds \leq \varphi(u_0).$$

Conversely, if $\frac{du}{dt} \in L^2(0, a; H)$ then from (3.13) we have

$$\varphi(u(t)) = \varphi(u(a)) + \int_t^a \|u'(s)\|^2 ds, \quad 0 < t < a,$$

and taking the limit $t \rightarrow 0$ shows

$$\varphi(u_0) \leq \varphi(u(a)) + \int_0^a \|u'(s)\|^2 ds$$

by lower-semi-continuity, and therefore from above follows (3.14). The remaining estimates follow easily. That is, from (3.14) and the general

$$\|u(t) - u_0\| \leq \int_0^t \|u'(s)\| ds \leq \sqrt{t} \left(\int_0^t \|u'(s)\|^2 ds \right)^{1/2}$$

follows the first, and from non-increase of $\|u'\|$ follows

$$t \|u'(t)\|^2 \leq \int_0^t \|u'(s)\|^2 ds$$

and, hence, the second. $\square$

IV.4. Additional Topics and Evolution Equations

We begin with some *Gronwall inequalities* that are frequently used in ordinary differential equations and that were implicit in some of our earlier estimates.

LEMMA 4.1. *Let $a(\cdot), b(\cdot) \in L^1(0, T)$ with $b(t) \geq 0$ a.e., and let the absolutely continuous $v : [0, T] \rightarrow \mathbb{R}^+$ satisfy*

$$(1 - \alpha)v'(t) \leq a(t)v(t) + b(t)v^\alpha(t), \quad \text{a.e. } t \in [0, T],$$

where $0 \leq \alpha < 1$. Then

$$v^{1-\alpha}(t) \leq v^{1-\alpha}(0)e^{\int_0^t a} + \int_0^t e^{\int_s^t a} b(s) ds, \quad 0 \leq t \leq T.$$

PROOF. Replace the non-negative $v(t)$ by $v(t) + \varepsilon$ for some $\varepsilon > 0$, and then divide by $(v(t) + \varepsilon)^\alpha$. Setting $w(t) = e^{-\int_0^t a}(v(t) + \varepsilon)^\alpha$, we have $w'(t) \leq e^{-\int_0^t a} b(t)$, and after integrating this inequality and letting $\varepsilon \rightarrow 0$, the desired estimate follows. $\square$

In the situation of Lemma 4.1 with $\alpha = 0$ we can integrate the assumed estimate to get a related weaker assumption which leads to the same resulting bound.

LEMMA 4.2. *Let $a \in L^1(0, T)$ and let $B : [0, T] \rightarrow \mathbb{R}$ be absolutely continuous; assume the bounded measurable function v satisfies*

$$v(t) \leq \int_0^t a(s)v(s) ds + B(t), \quad 0 \leq t \leq T.$$

Then we have

$$v(t) \leq B(0)e^{\int_0^t a} + \int_0^t e^{\int_s^t a} B'(s) ds, \quad 0 \leq t \leq T.$$

PROOF. Define $w(t) \equiv \int_0^t a(s)v(s) ds + B(t)$ and note that for a.e. $t \in [0, T]$

$$w'(t) \leq a(t)w(t) + B'(t),$$

so Lemma 4.1 applies with $\alpha = 0$. $\square$

Alternatively one can define $w(t) = \int_0^t a(s)v(s) ds$ and note that

$$w'(t) \leq a(t)w(t) + a(t)B(t).$$

Then Lemma 4.1 applies to give the following.

LEMMA 4.2'. *Let $a \in L^1(0, T)$ and $B \in L^\infty(0, T)$. Assume the bounded measurable function v satisfies a.e.*

$$v(t) \leq \int_0^t a(s)v(s) ds + B(t), \quad 0 \leq t \leq T.$$

Then we have

$$v(t) \leq B(t) + \int_0^t e^{\int_s^t a} a(s)B(s) ds, \quad 0 \leq t \leq T.$$

Note that the conclusions in Lemma 4.2 and Lemma 4.2' are equivalent; this follows from an integration-by-parts.

Next we show that by modifying the proof of Proposition 3.1 we obtain the corresponding results for the more general equation

$$(4.1) \quad \frac{du}{dt}(t) + A(u(t)) \ni \omega u(t) + f(t) .$$

THEOREM 4.1 (KATO). *Let A be m -accretive in the Hilbert space H and $\omega \geq 0$. For each $u_0 \in D(A)$ and absolutely continuous $f : [0, T] \rightarrow H$, there is a unique absolutely continuous $u : [0, T] \rightarrow H$, such that $u(0) = u_0$ and (4.1) holds at a.e. $t > 0$. Also, u is Lipschitz continuous and right-differentiable with $u(t) \in D(A)$ at every $t \geq 0$.*

PROOF. For any two solutions u_1, u_2 of (4.1) we have

$$\frac{1}{2} \frac{d}{dt} \|u_1(t) - u_2(t)\|^2 \leq \omega \|u_1(t) - u_2(t)\|^2 , \quad t \geq 0 ,$$

since A is accretive, and from here follows from Lemma 4.1 with $\alpha = 0$

$$\|u_1(t) - u_2(t)\| \leq e^{\omega t} \|u_1(0) - u_2(0)\| , \quad t \geq 0 .$$

Uniqueness is now immediate from the initial condition.

To obtain existence, let u_α be the unique solution for each $\alpha > 0$ of

$$(4.2) \quad \frac{du_\alpha}{dt}(t) + A_\alpha(u_\alpha(t)) = \omega u_\alpha(t) + f(t) , \quad 0 \leq t \leq T ,$$

with $u_\alpha(0) = u_0$. If $h > 0$, then $u_\alpha(t+h)$ is a solution of (4.2) with $f(t)$ replaced by $f(t+h)$, and the accretive estimate on A_α gives

$$\frac{1}{2} \frac{d}{dt} \|u_\alpha(t+h) - u_\alpha(t)\|^2 \leq \omega \|u_\alpha(t+h) - u_\alpha(t)\|^2 + \|f(t+h) - f(t)\| \|u_\alpha(t+h) - u_\alpha(t)\| .$$

Applying Lemma 4.1 with $\alpha = \frac{1}{2}$ to the preceding estimate and letting $h \rightarrow 0$ give

$$(4.3) \quad \|u'_\alpha(t)\| \leq e^{\omega t} \| -A_\alpha(u_0) + \omega u_0 + f(0) \| + \int_0^t e^{\omega(t-s)} \|f'(s)\| ds .$$

Since $\|A_\alpha(u_0)\| \leq \|A^0(u_0)\|$, it follows with (4.2) that u'_α, u_α , and $A_\alpha(u_\alpha)$ are bounded in $C([0, T], H)$.

To show $\{u_\alpha\}$ is Cauchy in $C([0, T], H)$, let $\alpha, \beta > 0$ and use (4.2) to obtain

$$\frac{1}{2} \frac{d}{dt} \|u_\alpha(t) - u_\beta(t)\|^2 = \omega \|u_\alpha(t) - u_\beta(t)\|^2 - (A_\alpha(u_\alpha(t)) - A_\beta(u_\beta(t)), u_\alpha(t) - u_\beta(t)) .$$

Writing $u_\alpha = \alpha A_\alpha u_\alpha + J_\alpha u_\alpha$ and similarly for u_β and using the accretive property of A , we obtain as in the proof of Proposition 3.1

$$\frac{1}{2} \frac{d}{dt} \|u_\alpha(t) - u_\beta(t)\|^2 \leq \omega \|u_\alpha(t) - u_\beta(t)\|^2 + \frac{\alpha + \beta}{4} K^2 ,$$

where $K \equiv \sup\{\|A_\alpha(u_\alpha(t))\| : 0 \leq t \leq T, \alpha > 0\}$. Thus we have

$$\|u_\alpha(t) - u_\beta(t)\|^2 \leq K^2 \frac{\alpha + \beta}{2} \cdot \frac{e^{2\omega t} - 1}{2\omega} , \quad 0 \leq t \leq T ,$$

so $\{u_\alpha\}$ is uniformly Cauchy and converges to a $u \in C([0, T], H)$ with

$$(4.4) \quad \|u_\alpha(t) - u(t)\|^2 \leq K^2 \cdot \frac{\alpha}{2} \cdot \frac{e^{2\omega t} - 1}{2\omega}, \quad 0 \leq t \leq T, \quad \alpha > 0;$$

from (4.3) it follows that $u' \in L^\infty(0, T; H)$. Also $u_\alpha - J_\alpha u_\alpha = \alpha A_\alpha(u_\alpha) \rightarrow 0$ in $C([0, T], H)$. Since $J_\alpha u_\alpha \rightarrow u$ and u'_α is bounded in $L^2(0, T; H)$, from Proposition 1.6 we obtain (4.1). The additional properties of this solution are verified just as in Proposition 3.1. $\square$

COROLLARY 4.1. *Let A be an operator on H such that for some $\omega_1 \geq 0$, $A + \omega_1 I$ is m -accretive. Let $B : \overline{D(A)} \rightarrow H$ be a Lipschitz function such that for some $\omega_2 > 0$*

$$\|B(u) - B(v)\| \leq \omega_2 \|u - v\|, \quad u, v \in \overline{D(A)}.$$

Then for each $\omega \geq 0$, $u_0 \in D(A)$ and absolutely continuous $f : [0, T] \rightarrow H$, there is a unique solution (as in Theorem 4.1) of

$$(4.5) \quad \frac{du}{dt}(t) + A(u(t)) + B(u(t)) \ni \omega u(t) + f(t)$$

with $u(0) = u_0$.

PROOF. It suffices to add $(\omega_1 + \omega_2)u(t)$ to both sides of (4.5), and then note that $B + \omega_2 I$ is accretive. Thus by Lemma 2.1 we are in the situation of Theorem 4.1 with ω replaced by $\omega + \omega_1 + \omega_2$. $\square$

We consider the problem of finding a *periodic* solution to the equation (4.1), that is,

$$(4.6.a) \quad \frac{du}{dt}(t) + A(u(t)) + \omega u(t) \ni f(t), \quad 0 < t < T,$$

$$(4.6.b) \quad u(0) = u(T).$$

This is equivalent to finding a periodic solution of (4.6.a) on all of $\mathbb{R}$ with period $T > 0$. Note that there is a solution of (4.6) with $A = 0$, $\omega = 0$ only if $\int_0^T f(t) dt = 0$, and in that case any two solutions differ by a constant. Thus, we expect existence or uniqueness only with additional positivity hypotheses on $A + \omega I$.

PROPOSITION 4.1. *Let A be m -accretive, $\omega > 0$, and $f : [0, T] \rightarrow H$ be absolutely continuous. Then there exists a unique absolutely continuous solution of (4.6).*

PROOF. We shall consider (4.6) in $\mathcal{H} = L^2(0, T; H)$. Thus, let $\mathcal{A}$ be the realization of A in $\mathcal{H}$ (see Example 2.C) and let $L = \frac{d}{dt}$ with domain $D(L) = \{u \in \mathcal{H} : u' \in \mathcal{H}, u(0) = u(T)\}$. Then $\mathcal{A}$ and L are m -accretive on $\mathcal{H}$ and the uniqueness of a solution follows immediately. In order to show the existence of a solution we consider the approximating problem (cf., (2.5))

$$(4.7) \quad Lu_\alpha + \mathcal{A}_\alpha(u_\alpha) + \omega u_\alpha = f$$

By Proposition 2.1 it suffices to show that $\{Lu_\alpha\} = \{u'_\alpha\}$ is bounded in $\mathcal{H}$. Extend each u_α and f to $\mathbb{R}$ as T -periodic and denote these extensions the same. If $h > 0$ we obtain from (4.7)

$$\frac{1}{2} \frac{d}{dt} \|u_\alpha(t+h) - u_\alpha(t)\|^2 + \omega \|u_\alpha(t+h) - u_\alpha(t)\|^2 \leq \|f(t+h) - f(t)\| \|u_\alpha(t+h) - u_\alpha(t)\|.$$

Thus for $0 \leq s < s + h \leq T$, $t = T + s$ we get from Lemma 4.1

$$e^{\omega t} \|u_\alpha(t + h) - u_\alpha(t)\| \leq e^{\omega s} \|u_\alpha(s + h) - u_\alpha(s)\| + \int_s^t e^{\omega \tau} \|f(\tau + h) - f(\tau)\| d\tau ,$$

and since u_α is T -periodic this gives

$$(4.8) \quad \begin{aligned} & \|u_\alpha(s + h) - u_\alpha(s)\| (1 - e^{-\omega T}) \\ & \leq \int_s^{T+s} \|f(\tau + h) - f(\tau)\| d\tau , \quad 0 \leq s < s + h \leq T . \end{aligned}$$

Denote the *variation* of f on $[0, t]$ by

$$V(t) = \sup \left\{ \sum_{j=0}^{n-1} \|f(t_{j+1}) - f(t_j)\| : 0 = t_0 < t_1 < \dots < t_n = t \right\} ,$$

where the supremum is taken over all partitions of $[0, t]$; then note

$$V(\tau) + \|f(\tau + h) - f(\tau)\| \leq V(\tau + h) , \quad \tau \geq 0 ,$$

so from (4.8) follows by monotonicity of V

$$\begin{aligned} \|u_\alpha(s + h) - u_\alpha(s)\| (1 - e^{-\omega T}) & \leq \int_s^{T+s} (V(\tau + h) - V(\tau)) d\tau \\ & \leq h(V(T + s + h) - V(s)) , \quad 0 \leq s < s + h \leq T . \end{aligned}$$

Finally we note that since f is absolutely continuous on $[0, T]$ and on $[T, 2T]$ we have

$$V(T + s + h) - V(s) \leq \int_s^T \|f'\| d\tau + \|f(T) - f(0)\| + \int_T^{T+s+h} \|f'\| d\tau ,$$

so we let $h \searrow 0$ above to obtain

$$(4.9) \quad \|u'_\alpha(s)\| \leq (1 - e^{-\omega T})^{-1} (\|f'\|_{L^1(0, T)} + \|f(0) - f(T)\|) , \quad \text{a.e. } s \in [0, T] .$$

Thus $\{u'_\alpha\}$ is bounded in $\mathcal{H}$ and Proposition 2.1 gives existence. The solution u of (4.6) satisfies (4.9), so it is Lipschitz continuous; it is a solution of the Cauchy problem on $[\tau, T]$ for a.e. $\tau \in [0, T]$, so it has the additional properties described in Theorem 4.1. $\square$

We obtained in Theorem 4.1 the existence of a solution of the Cauchy problem

$$(4.10) \quad \frac{du}{dt} + A(u) \ni f + \omega u , \quad u(0) = u_0 ,$$

whenever f is absolutely continuous and $u_0 \in D(A)$. Moreover it follows that if u_1, u_2 are solutions of respective Cauchy problems with data f_1, f_2 in $L^1(0, T; H)$ and u_0^1, u_0^2 in $\overline{D(A)}$, then from accretivity of A follows

$$\frac{1}{2} \frac{d}{dt} e^{-2\omega t} \|u_1(t) - u_2(t)\|^2 \leq e^{-2\omega t} (f_1(t) - f_2(t), u_1(t) - u_2(t)) .$$

This leads directly to

$$(4.11) \quad \frac{1}{2}e^{-2\omega t}\|u_1(t) - u_2(t)\|^2 \leq \frac{1}{2}e^{-2\omega s}\|u_1(s) - u_2(s)\|^2 + \int_s^t e^{-2\omega\tau}(f_1 - f_2, u_1 - u_2) d\tau, \quad 0 \leq s \leq t \leq T,$$

and with Lemma 4.1 it also leads to

$$(4.12) \quad e^{-\omega t}\|u_1(t) - u_2(t)\| \leq e^{-\omega s}\|u_1(s) - u_2(s)\| + \int_s^t e^{-\omega\tau}\|f_1(\tau) - f_2(\tau)\| d\tau, \quad 0 \leq s \leq t \leq T.$$

The estimate (4.12) with $s = 0$ leads to the following notion.

DEFINITION. A *generalized solution* of (4.1) is a function $u \in C([0, T], H)$ for which there exists a sequence of (absolutely continuous) solutions u_n of

$$\frac{du_n}{dt} + A(u_n) \ni f_n + \omega u_n, n \geq 1$$

with $f_n \rightarrow f$ in $L^1(0, T; H)$ and $u_n \rightarrow u$ in $C([0, T], H)$.

Since (4.11) and (4.12) are stable with respect to such limits, they hold also for generalized solutions, and thus we obtain uniqueness for the Cauchy problem (4.10). Also, we obtain from Theorem 4.1 the following immediate extension.

THEOREM 4.1A. *If A is m -accretive on H , $\omega \geq 0$, $f \in L^1(0, T; H)$ and $u_0 \in \overline{D(A)}$, then there is a unique generalized solution of (4.10) on $[0, T]$, and any two generalized solutions of respective problems satisfy the estimates (4.11) and (4.12).*

The existence follows by approximating the given f and u_0 by a sequence of absolutely continuous $\{f_n\}$ and a sequence $\{u_0^n\}$ in $D(A)$, respectively. The estimates (4.11) and (4.12) hold in the limit as indicated above.

By the same proof we obtain a generalized solution of the Cauchy problem for (4.5) with the relaxed assumptions on u_0 and f .

COROLLARY 4.1A. *With A, B given as in Corollary 1, for each $u_0 \in \overline{D(A)}$ and $f \in L^1(0, T; H)$ there is a unique generalized solution of (4.5) with $u(0) = u_0$.*

Moreover, one obtains a generalized solution of the periodic problem (4.6) for this wider class of functions, f .

PROPOSITION 4.1A. *Let A be m -accretive, $\omega > 0$, and $f \in L^1(0, T; H)$. Then there exists a unique generalized solution of (4.6).*

PROOF. This follows directly from Proposition 4.1 by approximating f with a sequence of absolutely continuous functions.

Alternatively, one can solve (4.6) directly by a fixed-point argument based on the periodicity condition (4.6.b). For each $u_0 \in \overline{D(A)}$ there is a unique generalized solution u of (4.6.a) with $u(0) = u_0$; define $\mathcal{F} : \overline{D(A)} \rightarrow \overline{D(A)}$ by $\mathcal{F}(u_0) = u(T)$. Thus, (4.6) is equivalent to $\mathcal{F}(u_0) = u_0$, and $\mathcal{F}$ is a strict contraction if $\omega > 0$. $\square$

The notion of a generalized solution of (4.1) is based directly on the estimate (4.12) which implies the solution of Cauchy problem (4.10) depends continuously on

the pair $[f, u_0]$ in $L^1(0, T; H) \times H$. We shall show the solution depends continuously on the triple $[A, f, u_0]$ with the appropriate notion of convergence of operators. We set $\omega = 0$ without loss of generality.

THEOREM 4.2 (BREZIS-PAZY-MIYADERA-OHARU). *Let A^n , $n \geq 1$, and A be m -accretive in the Hilbert space H , let f_n and f in $L^1(0, T; H)$, $u_0^n \in \overline{D(A^n)}$ and $u_0 \in \overline{D(A)}$ be given. Let u^n, u be the respective generalized solutions of*

$$\begin{aligned} \frac{du^n}{dt} + A^n(u^n) &\ni f_n, & u^n(0) &= u_0^n, \\ \frac{du}{dt} + A(u) &\ni f, & u(0) &= u_0. \end{aligned}$$

If $u_0^n \rightarrow u_0$ in H , $f_n \rightarrow f$ in $L^1(0, T; H)$, and if

$$(4.13) \quad (I + \alpha A^n)^{-1} w \rightarrow (I + \alpha A)^{-1} w, \quad w \in H, \alpha > 0,$$

then $u^n \rightarrow u$ in $C([0, T], H)$.

PROOF. We shall proceed as follows. For the case of $f_n = f = 0$, we regularize the initial data to get nearby (absolutely continuous) solutions, regularize the operator to get the Lipschitz case, then estimate directly the Lipschitz case with the Gronwall type inequalities. The corresponding forced equations, with f_n and f as given, are thereafter easily obtained.

Consider the case $f_n = f = 0$ and let $u, v_\alpha, w_\alpha, w_\alpha^n, v_\alpha^n, u^n$ be the generalized solutions of the following Cauchy problems with $n \geq 1, \alpha > 0$,

$$\begin{aligned} \frac{du}{dt} + A(u) &\ni 0, & u(0) &= u_0 \in \overline{D(A)}, \\ \frac{dv_\alpha}{dt} + A(v_\alpha) &\ni 0, & v_\alpha(0) &= (I + \sqrt{\alpha}A)^{-1}u_0 \in D(A), \\ \frac{dw_\alpha}{dt} + A_\alpha(w_\alpha) &= 0, & w_\alpha(0) &= (I + \sqrt{\alpha}A)^{-1}u_0, \\ \frac{dw_\alpha^n}{dt} + A_\alpha^n(w_\alpha^n) &= 0, & w_\alpha^n(0) &= (I + \sqrt{\alpha}A^n)^{-1}u_0^n, \\ \frac{dv_\alpha^n}{dt} + A^n(v_\alpha^n) &\ni 0, & v_\alpha^n(0) &= (I + \sqrt{\alpha}A^n)^{-1}u_0^n \in D(A^n), \\ \frac{du^n}{dt} + A^n(u^n) &\ni 0, & u^n(0) &= u_0^n \in \overline{D(A^n)}. \end{aligned}$$

Since A and A^n are accretive we obtain from (4.12)

$$\begin{aligned} \|u - v_\alpha\|_{C([0, T], H)} &\leq \|u_0 - J_{\sqrt{\alpha}}u_0\|_H, \\ \|u^n - v_\alpha^n\|_{C([0, T], H)} &\leq \|u_0^n - J_{\sqrt{\alpha}}^n u_0^n\|_H. \end{aligned}$$

Using the convergence-rate estimate from the proof of Proposition 3.1, and then observing $A_{\sqrt{\alpha}} \subset A \circ J_{\sqrt{\alpha}}$, one finds

$$\begin{aligned} \|v_\alpha - w_\alpha\|_{C([0, T], H)} &\leq \left(\frac{\alpha T}{2}\right)^{1/2} \|A^0(J_{\sqrt{\alpha}}u_0\| \\ &\leq \left(\frac{\alpha T}{2}\right)^{1/2} \left\| \frac{1}{\sqrt{\alpha}}(I - J_{\sqrt{\alpha}})u_0 \right\|, \end{aligned}$$

and thus we have

$$\begin{aligned}\|v_\alpha - w_\alpha\|_{C([0,T],H)} &\leq \left(\frac{T}{2}\right)^{1/2} \|u_0 - J_{\sqrt{\alpha}}u_0\| , \\ \|v_\alpha^n - w_\alpha^n\|_{C([0,T],H)} &\leq \left(\frac{T}{2}\right)^{1/2} \|u_0^n - J_{\sqrt{\alpha}}u_0^n\| .\end{aligned}$$

In order to estimate the remaining link, $w_\alpha - w_\alpha^n$, we subtract their corresponding equations, integrate over the interval $[0, t]$, and then use the Lipschitz continuity of A_α^n to obtain

$$\begin{aligned}\|w_\alpha(t) - w_\alpha^n(t)\| &\leq \|w_\alpha(0) - w_\alpha^n(0)\| + \int_0^t \|A_\alpha(w_\alpha) - A_\alpha^n(w_\alpha^n)\| ds \\ &\leq \|w_\alpha(0) - w_\alpha^n(0)\| + \int_0^T \|A_\alpha(w_\alpha) - A_\alpha^n(w_\alpha)\| ds \\ &\quad + \frac{1}{\alpha} \int_0^t \|w_\alpha(s) - w_\alpha^n(s)\| ds .\end{aligned}$$

From Lemma 4.2 and the definition of A_α , A_α^n we get

$$\|w_\alpha - w_\alpha^n\|_{C([0,T],H)} \leq \left(\|J_{\sqrt{\alpha}}u_0 - J_{\sqrt{\alpha}}u_0^n\| + \frac{1}{\alpha} \|J_\alpha w_\alpha - J_\alpha^n w_\alpha\|_{L^1(0,T;H)} \right) e^{T/\alpha} .$$

From (4.13) and the Lebesgue Theorem it follows that $\|J_\alpha w_\alpha - J_\alpha^n w_\alpha\|_{L^1(0,T;H)} \rightarrow 0$ as $n \rightarrow \infty$. In summary, we have for every $\alpha > 0$,

$$\limsup_{n \rightarrow \infty} \|u - u^n\|_{C([0,T],H)} \leq 2 \left(1 + \left(\frac{T}{2}\right)^{1/2} \right) \|u_0 - J_{\sqrt{\alpha}}u_0\| ,$$

so it follows that $u^n \rightarrow u$ in $C([0, T], H)$.

For the case of $f_n = f = \xi$, a constant vector in H , the result follows directly from above by considering the translated operators $A(u) - \xi$, $A^n(u) - \xi$; one need only check that (4.13) holds. Thus the result holds also for the case $f_n = f = g$, where g is a piece-wise constant function in $L^1(0, T; H)$. If v, v^n are the generalized solutions of

$$\begin{aligned}\frac{dv}{dt} + A(v) &\ni g , & v(0) &= u_0 , \\ \frac{dv^n}{dt} + A^n(v^n) &\ni g , & v^n(0) &= u_0^n ,\end{aligned}$$

then from (4.13) follows

$$\begin{aligned}\|u - u^n\|_{C([0,T],H)} &\leq \|u - v\| + \|v - v^n\| + \|v^n - u^n\| \\ &\leq \|f - g\|_{L^1(0,T;H)} + \|v - v^n\|_{C([0,T],H)} + \|g - f_n\|_{L^1(0,T;H)} ,\end{aligned}$$

so we have

$$\limsup_{n \rightarrow \infty} \|u - u^n\|_{C([0,T],H)} \leq 2\|f - g\|$$

for every piece-wise constant function g . But these are dense in $L^1(0, T; H)$, so the desired result follows. $\square$

We consider next those m -accretive operators which are subgradients and the effects of the *regularizing property* of (4.1). From Theorem 4.1A we know there is

a unique generalized solution u of (4.1) when $u_0 \in \overline{D(A)}$ and $f \in L^1(0, T; H)$. It follows from Theorem 4.1 that if $u_0 \in D(A)$ and f is absolutely continuous, then u is absolutely continuous. However, Corollary 3.2 shows that for the case $f = 0$, if A is a subgradient, the generalized solution is *always* absolutely continuous. We shall extend this last result to the equation (4.1) with $f \in L^2(0, T; H)$.

In the proof of Corollary 3.2 we made our estimates on the smooth approximations to the operator. In order to estimate directly on the equation with general data, we begin with the following *chain rule*.

LEMMA 4.3. *Let $\varphi : H \rightarrow \mathbb{R}_\infty$ be proper, convex, and lower-semi-continuous on the Hilbert space H . Denote the subgradient by $\partial\varphi$. If $u, \frac{du}{dt} \in L^2(0, T; H)$ and if there exists a $g \in L^2(0, T; H)$ with $g \in \partial\varphi(u)$ a.e. on $[0, T]$, then the function $\varphi \circ u$ is absolutely continuous on $[0, T]$ and*

$$\frac{d}{dt}\varphi(u(t)) = \left(h(t), \frac{du}{dt}(t) \right), \quad \text{a.e. } t \in [0, T]$$

for any function h with $h \in \partial\varphi(u)$ a.e. on $[0, T]$.

PROOF. Set $A = \partial\varphi$ and recall from Proposition 1.9 that $A_\alpha = \varphi'_\alpha$, the Frechet derivative of the convex φ_α , for each $\alpha > 0$. By the usual chain rule we have

$$\frac{d}{dt}\varphi_\alpha(u(t)) = \left(A_\alpha(u(t)), \frac{du}{dt}(t) \right), \quad \text{a.e. } t \in [0, T],$$

so there follows

$$\varphi_\alpha(u(t)) - \varphi_\alpha(u(s)) = \int_s^t \left(A_\alpha(u), \frac{du}{dt} \right) d\tau, \quad s, t \in [0, T].$$

Letting $\alpha \rightarrow 0$ yields

$$\varphi(u(t)) - \varphi(u(s)) = \int_s^t \left(A^0(u), \frac{du}{dt} \right) d\tau, \quad s, t \in [0, T],$$

since the existence of the function g as above shows the integrand belongs to $L^1(0, T; H)$, so $\varphi \circ u$ is absolutely continuous. Pick a $t \in (0, T)$ for which $u(t) \in D(A)$ and $u'(t)$, $(\varphi \circ u)'(t)$ exist. For any vector $\xi \in \partial\varphi(u(t))$ we have

$$(\xi, v - u(t)) \leq \varphi(v) - \varphi(u(t)), \quad v \in H.$$

By setting $v = u(t \pm \varepsilon)$, dividing by ε , and letting $\varepsilon \rightarrow 0$, we obtain

$$(\xi, u'(t)) = \frac{d}{dt}\varphi(u(t))$$

as desired. □

THEOREM 4.3 (BREZIS). *Let $\varphi : H \rightarrow \mathbb{R}_\infty$ be proper, convex, and lower-semi-continuous on the Hilbert space H and set $A = \partial\varphi$. Let u be the generalized solution of*

$$\frac{du}{dt} + A(u) \ni f \text{ a.e. on } [0, T], \quad u(0) = u_0$$

with $f \in L^2(0, T; H)$ and $u_0 \in \overline{D(A)}$. Then

$$\varphi \circ u \in L^1(0, T), \quad \sqrt{t} \frac{du}{dt} \in L^2(0, T; H), \quad \text{and } u(t) \in D(A), \text{ a.e. } t \in [0, T].$$

If in addition $u_0 \in \text{dom}(\varphi)$, then

$$\varphi \circ u \in L^\infty(0, T) \quad , \quad \frac{du}{dt} \in L^2(0, T; H) \quad .$$

PROOF. We may assume without loss of generality that

$$\varphi(x_0) = 0 = \min\{\varphi(x) : x \in H\} \quad .$$

Otherwise, pick any $[x_0, y_0] \in \partial\varphi$ and set $\psi(u) = \varphi(u) - \varphi(x_0) - \langle y_0, u - x_0 \rangle$. Then our equation is equivalent to $\frac{du}{dt} + \partial\psi(u) \ni f - y_0$.

We proceed as in Proposition 3.2. At first we assume $u_0 \in D(A)$ and $f' \in L^2(0, T; H)$. Since $f - u' \in \partial\varphi(u)$ we have

$$\varphi(u(t)) \leq (f(t) - u'(t), u(t) - x_0) \quad , \quad \text{a.e. } t \in [0, T] \quad .$$

so we obtain

$$\int_0^T \varphi(u(t)) \, dt \leq \frac{1}{2} \|u_0 - x_0\|^2 + \int_0^T \|f(t)\| \|u(t) - x_0\| \, dt \quad .$$

Also, by applying $u(t) - x_0$ to our evolution equation and using Lemma 4.1 we get

$$\|u(t) - x_0\| \leq \|u_0 - x_0\| + \int_0^t \|f(s)\| \, ds$$

so this gives with the above

$$(4.14) \quad \int_0^T \varphi(u(t)) \, dt \leq \left(\|u_0 - x_0\| + \int_0^T \|f\| \, dt \right)^2 \quad .$$

This is the *energy estimate*. Next we take the scalar product of our equation with $tu'(t)$ and use Lemma 4.3 to obtain

$$t\|u'(t)\|^2 + t \frac{d}{dt} \varphi(u(t)) = t(f(t), u'(t)) \quad ,$$

and therefore

$$\int_0^T t\|u'(t)\|^2 \, dt + T\varphi(u(T)) = \int_0^T t(f(t), u'(t)) \, dt + \int_0^T \varphi(u(t)) \, dt \quad .$$

Since φ is non-negative this leads to the *rate estimate*

$$(4.15) \quad \int_0^T t\|u'(t)\|^2 \, dt \leq \int_0^T t\|f(t)\|^2 \, dt + 2 \int_0^T \varphi(u(t)) \, dt \quad .$$

By combining (4.14) and (4.15) we have

$$(4.16) \quad \int_0^T t\|u'(t)\|^2 \, dt \leq \int_0^T t\|f(t)\|^2 \, dt + 2 \left(\|u_0 - x_0\| + \int_0^T \|f(t)\| \, dt \right)^2 \quad .$$

Assume $u_0 \in \overline{D(A)}$ and that $f \in L^2(0, T; H)$. Let $\{u_0^n\}$ be a sequence in $D(A)$ with $u_0^n \rightarrow u_0$ in H and let $\{f_n\}$ be a sequence with each $f_n' \in L^2(0, T; H)$ and with $f_n \rightarrow f$ in $L^2(0, T; H)$. Then it follows from (4.12) that the solutions u^n corresponding to the data u_0^n, f_n , converge uniformly to u on $[0, T]$. From (4.16) applied to $\{u^n\}$ it follows that $\{\sqrt{t} \frac{du^n}{dt}\}$ converges weakly in $L^2(0, T; H)$ to $\sqrt{t} \frac{du}{dt}$. Thus u is absolutely continuous on each $[\delta, T]$, $\delta > 0$, and we show as before that

u satisfies the equation (4.1) with $\omega = 0$ in each $L^2(\delta, T; H)$ since $A = \partial\varphi$ is m -accretive in that space.

Finally we assume $u_0 \in \text{dom}(\varphi)$. Take the scalar product of the equation with $u'(t)$ and use Lemma 4.3 to obtain

$$\|u'(t)\|^2 + \frac{d}{dt}\varphi(u(t)) \leq \|f(t)\| \|u'(t)\| ,$$

and therefore at a.e. $t \in [0, T]$

$$\frac{1}{2}\|u'(t)\|^2 + \frac{d}{dt}\varphi(u(t)) \leq \frac{1}{2}\|f(t)\|^2 .$$

By integrating over $[s, t] \subset (0, T]$ and then letting $s \rightarrow 0^+$ we find

$$\max\left\{\frac{1}{2}\|u'\|_{L^2(0, T; H)} , \|\varphi \circ u\|_{C([0, T], H)}\right\} \leq \varphi(u_0) + \frac{1}{2}\|f\|_{L^2(0, T; H)} ,$$

and this finishes the proof. $\square$

We consider finally the *singular-in-time* evolution equation

$$(4.17) \quad \frac{1}{b(\tau)} \frac{dv(\tau)}{d\tau} + \omega v(\tau) + A(v(\tau)) \ni g(\tau) , \quad -\infty \leq \tau_0 < \tau < \tau_1 ,$$

where b is a locally-integrable positive function on the interval (τ_0, τ_1) . The effect of the singularity at τ_0 and at τ_1 can be determined from the change-of-variable $t = B(\tau)$ where B is absolutely continuous with $B'(\tau) = b(\tau)$ a.e. on (τ_0, τ_1) . Thus v is a solution of (4.17) if and only if the function $u = v \circ B^{-1}$ satisfies

$$(4.18) \quad \frac{du(t)}{dt} + \omega u(t) + A(u(t)) \ni f(t)$$

on the interval $(B(\tau_0), B(\tau_1))$ with $f = g \circ B^{-1}$. If b is *weakly singular* at τ_0 , i.e., if $b(\cdot)$ is integrable at τ_0 , so $B(\tau_0) > -\infty$, then the Cauchy problem is appropriate for (4.17) and one obtains a well-posed problem by specifying initially $v(\tau_0) = v_0$. But if b is *strongly singular* at τ_0 , so that $B(\tau_0) = -\infty$, then we are led to a *history-value problem*: find a solution of (4.18) on $(-\infty, 0]$. Of course one can use the preceding results on the Cauchy problem to uniquely extend this solution to all of $(-\infty, B(\tau_1))$.

We shall work in the Hilbert space $\mathcal{H} \equiv L^2(-\infty, 0; H)$. Let L be the operator given by $Lu = \frac{du}{dt}$ on the domain $D(L) = \{u \in \mathcal{H} : \frac{du}{dt} \in \mathcal{H}\}$.

LEMMA 4.4. L is m -accretive on $\mathcal{H}$ and for each $u \in D(L)$, $\lim_{t \rightarrow -\infty} u(t) = 0$. For each $\alpha > 0$, $u + \alpha Lu = f$ if and only if

$$(4.19) \quad u(t) = \int_{-\infty}^t f(s) \frac{1}{\alpha} e^{s-t/\alpha} ds , \quad -\infty < t < 0 .$$

PROOF. If $(I + \alpha L)u = f$, then

$$e^{t/\alpha} u(t) - e^{s/\alpha} u(s) = \int_s^t f(r) \frac{1}{\alpha} e^{r/\alpha} dr , \quad s < t \leq 0 ,$$

and this gives

$$\|e^{t/\alpha}u(t) - e^{s/\alpha}u(s)\|^2 \leq \int_s^t \|f\|^2 dr \frac{1}{2\alpha} (e^{2t/\alpha} - e^{2s/\alpha}) .$$

This shows that $\lim_{t \rightarrow -\infty} e^{t/\alpha}u(t) \equiv h$ exists in H and that

$$u(t) = he^{-t/\alpha} + \int_{-\infty}^t f(s) \frac{1}{\alpha} e^{s-t/\alpha} ds , \quad t \leq 0 .$$

We shall see below that the second term belongs to $\mathcal{H}$; the first is in $\mathcal{H}$ only if $h = 0$, so it follows that $u(-\infty) = 0$.

Next let $f \in \mathcal{H}$, $\alpha > 0$, and define u by (4.19). The integral converges because $e^{t/\alpha} \in \mathcal{H}$. Note also that

$$\int_{-\infty}^t \frac{1}{\alpha} e^{s-t/\alpha} ds = 1$$

so the convexity of the norm-squared shows that

$$(4.20) \quad \|u(t)\|^2 \leq \int_{-\infty}^t \|f(s)\|^2 \frac{1}{\alpha} e^{s-t/\alpha} ds .$$

From Fubini's Theorem we get

$$\|u\|_{\mathcal{H}}^2 \leq \int_{-\infty}^0 \int_s^0 \|f(s)\|^2 \frac{1}{\alpha} e^{s-t/\alpha} dt ds \leq \|f\|_{\mathcal{H}}^2 ,$$

so $u \in D(L)$ and L is m -accretive. □

COROLLARY 4.2. *For each $u \in D(L)$, $\alpha > 0$,*

$$\sup\{\|u(t)\| : t \leq 0\} \leq \frac{1}{\sqrt{2\alpha}} \|(I + \alpha L)u\|_{\mathcal{H}} .$$

PROOF. This is immediate from (4.19). □

PROPOSITION 4.2. *Let A be m -accretive on H , $A(0) \ni 0$, and $\omega > 0$. Then for each $f \in D(L)$ there exists a unique $u \in D(L)$ for which (4.18) holds at a.e. $t \leq 0$, that is, $Lu + \omega u + A(u) \ni f$ in $\mathcal{H}$.*

PROOF. Let A_α be the accretive Lipschitz approximation of A and consider for each $\alpha > 0$ the solution u_α (guaranteed by Lemma 2.1) of

$$(4.21) \quad Lu_\alpha + \omega u_\alpha + A_\alpha(u_\alpha) = f .$$

From Proposition 2.1 and Lemma 2.2 it follows that (4.18) has a solution if (and only if) $\{Lu_\alpha\}$ is bounded in $\mathcal{H}$. To verify this, let $h > 0$ and extend f and the solutions u_α to $(-\infty, h)$. By the accretive estimate on A_α we obtain successively

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u_\alpha(t+h) - u_\alpha(t)\|^2 + \omega \|u_\alpha(t+h) - u_\alpha(t)\|^2 \\ & \leq (f(t+h) - f(t), u_\alpha(t+h) - u_\alpha(t))_H \\ & \leq \frac{1}{2} \left\{ \frac{1}{\omega} \|f(t+h) - f(t)\|^2 + \omega \|u_\alpha(t+h) - u_\alpha(t)\|^2 \right\} , \end{aligned}$$

and then

$$\|u_\alpha(t+h) - u_\alpha(t)\|^2 e^{\omega t} \leq \|u_\alpha(\tau+h) - u_\alpha(\tau)\|^2 e^{\omega \tau} + \frac{1}{\omega} \int_\tau^t \|f(s+h) - f(s)\|^2 e^{\omega s} ds ,$$

Since $\lim_{\tau \rightarrow -\infty} (u_\alpha(\tau) e^{\omega \tau}) = 0$, we get

$$\|u_\alpha(t+h) - u_\alpha(t)\|^2 \leq \frac{1}{\omega} \int_{-\infty}^t \|f(s+h) - f(s)\|^2 e^{\omega(s-t)} ds ,$$

and then by Fubini's Theorem

$$\begin{aligned} \int_{-\infty}^0 \|u_\alpha(t+h) - u_\alpha(t)\|^2 dt &\leq \frac{1}{\omega} \int_{-\infty}^0 \|f(s+h) - f(s)\| \int_s^0 e^{\omega(s-t)} dt ds \\ &\leq \frac{1}{\omega^2} \int_{-\infty}^0 \|f(s+h) - f(s)\|^2 ds . \end{aligned}$$

After dividing by h^2 and letting $h \rightarrow 0$ we have

$$\left\| \frac{du_\alpha}{dt} \right\|_{\mathcal{H}} \leq \frac{1}{\omega} \left\| \frac{df}{dt} \right\|_{\mathcal{H}} , \quad \alpha > 0 ,$$

and this completes the proof. $\square$

When the operator A is a subgradient, we can as usual relax the hypotheses on the data.

PROPOSITION 4.3. *Let $\varphi : H \rightarrow \mathbb{R}_\infty$ be convex, lower-semi-continuous, and $\varphi(0) = 0 = \min\{\varphi(x) : x \in H\}$. Then for each $\omega > 0$ and $f \in \mathcal{H}$ there is a unique $u \in D(L)$ for which (4.18) holds at a.e. $t \leq 0$.*

PROOF. We shall apply Proposition 2.2 to show $L + \partial\Phi$ is m -accretive, where

$$\Phi(u) \equiv \int_{-\infty}^0 \varphi(u(t)) dt , \quad u \in \mathcal{H} .$$

Let $u \in \mathcal{H}$ and $(I + \alpha L)u_\alpha = u$, $\alpha > 0$. Using the representation (4.19) and the convexity of φ we get

$$\varphi(u_\alpha(t)) \leq \int_{-\infty}^t \varphi(u(s)) \frac{1}{\alpha} e^{\frac{s-t}{\alpha}} ds .$$

From this follows

$$\Phi(u_\alpha) \leq \int_{-\infty}^0 \varphi(u(s)) \int_s^0 \frac{1}{\alpha} e^{\frac{s-t}{\alpha}} dt ds$$

by Fubini, and hence $\Phi(u_\alpha) \leq \Phi(u)$ as desired. $\square$

IV.5. Parabolic Equations and Inequalities

We shall apply the preceding results on the solvability of the abstract Cauchy problem (4.10) to the case of operators A constructed from monotone operators which correspond to elliptic boundary-value problems. Many such examples were given in II.5 and we shall refer to them in the following. As we have seen above, sharper results are possible for the class of subgradient operators, and we briefly explore how these are obtained from corresponding subdifferentials; see II.8 and IV.2 for examples. The results of this section should be compared with those of III.4; there we resolved the Cauchy problems in somewhat more generality, whereas, here we shall obtain solutions which are more regular.

Assume V is a reflexive and separable Banach space which is dense and continuously imbedded in a Hilbert space H . We identify H with its dual H' by the Riesz map and thereby identify H as a subspace of V' with

$$f(v) = (f, v)_H, \quad f \in H, \quad v \in V.$$

Assume we are given a function $\mathcal{A} : V \rightarrow V'$ which is monotone and demicontinuous. Then define an operator A on H with domain $D(A) = \{u \in V : \mathcal{A}(u) \in H\}$ and values $Au = \mathcal{A}(u)$, $u \in D(A)$. Since

$$(Au - Av, u - v)_H = \langle \mathcal{A}u - \mathcal{A}v, u - v \rangle, \quad u, v \in D(A),$$

and since $\mathcal{A}$ is monotone, it follows that A is accretive on H . We define an operator A on H to be *regular accretive* if it is so constructed from $\mathcal{A}$, V and H as above.

LEMMA 5.1. *Let A be regular accretive on H and assume*

$$(5.1) \quad \lim_{\|v\| \rightarrow +\infty} \frac{|v|_H^2 + \mathcal{A}v(v)}{|v|_H} = +\infty.$$

Then A is m -accretive.

PROOF. Let $f \in H$. In order to show that $f \in Rg(I + A)$ it suffices by Theorem II.2.2 to show there is a $\rho > 0$ such that $(I + \mathcal{A})v(v) > (f, v)_H$ for all $v \in V$ with $\|v\| > \rho$. Suppose on the contrary that $v \in V$ with

$$|v|_H^2 + \mathcal{A}v(v) \leq |f|_H |v|_H.$$

That is, $(|v|_H^2 + \mathcal{A}v(v))/|v|_H \leq |f|_H$. Then (5.1) shows there is a ρ for which $\|v\| \leq \rho$. Thus, there is a $u \in V : u + \mathcal{A}u = f$, and it is clear that $u \in D(A)$, so $(I + A)u = f$. $\square$

REMARK. Assume there is a seminorm $[\cdot]$ on V and numbers $\lambda > 0$, $\alpha > 0$ such that

$$[v] + \lambda |v|_H \geq \alpha \|v\|, \quad \text{and} \\ \mathcal{A}v(v) \geq \alpha [v]^p - \lambda |v|_H, \quad v \in V.$$

Then (5.1) holds. Recall that these assumptions were used in Proposition III.4.1 and Corollary III.4.1 to resolve the Cauchy problem. Lemma 5.1 leads to the following result which gives a more regular solution when the data is appropriately more regular.

PROPOSITION 5.1. *If A is regular accretive and (5.1) holds, then for each $u_0 \in D(A)$ and $f \in W^{1,1}(0, T; H)$ there exists a unique $u \in W^{1,\infty}(0, T; H)$ with $u(t) \in D(A)$ for all $0 \leq t \leq T$, $u(0) = u_0$, and*

$$(5.2) \quad \frac{du}{dt}(t) + A(u(t)) = f(t) , \quad \text{a.e. } t \in (0, T) .$$

Also, the right-derivative satisfies

$$D^+u(t) + A(u(t)) = f(t) , \quad t \in [0, T) .$$

PROOF. This follows directly from Lemma 5.1 and Theorem 4.1. $\square$

It is instructive to compare Proposition 5.1 with Proposition III.4.1. In the latter we are permitted a time-dependent family of operators, but with an explicit uniform growth rate from V to V' . With more general data, $f \in L^{p'}(0, T; V')$ and $u_0 \in H$, we obtain from Proposition III.4.1 a solution $u \in L^p(0, T; V)$ for which the equation (5.2) holds in $L^{p'}(0, T; V)$. By comparison, we need for Proposition 5.1 more restrictions on the data but we obtain a solution u satisfying stronger properties; in particular, the equation (5.2) holds in H . The requirement that $u(t) \in D(A)$ is specifically related to boundary conditions when $\mathcal{A}$ corresponds to an elliptic boundary-value problem.

For some examples of initial-boundary-value problems we recall the following model problem from III.4. Let G be a bounded domain in $\mathbb{R}^n$ whose boundary ∂G is a C^1 manifold, $2 \leq p < \infty$, for simplicity take $a(\xi) = |\xi|^{p-1} \text{sgn } \xi$, and let $b \in L^\infty(G)$, and $c \in L^\infty(\partial G)$ be non-negative. Let V be a closed subspace of $W^{1,p}(G)$ and define

$$(5.3) \quad \begin{aligned} \mathcal{A}u(v) = & \int_G \left\{ \sum_{j=1}^n a(\partial_j u(x)) \partial_j v(x) + b(x) a(u(x)) v(x) \right\} dx \\ & + \int_{\partial G} c(s) a(\gamma u(s)) \gamma v(s) ds , \quad u, v \in V . \end{aligned}$$

From Lemma 5.1, its following Remark, and II.5, we find that A is m -accretive on $H = L^2(G)$. Moreover, $\mathcal{A}u = f \in L^2(G)$ if and only if

$$u \in V : \mathcal{A}u(v) = \int_G f v dx , \quad v \in V .$$

We shall assume $C_0^\infty(G) \subset V$, so this implies

$$(5.4.a) \quad A(u) = - \sum_{j=1}^n \partial_j a(\partial_j u) + ba(u) \quad \text{in } \mathcal{D}^* .$$

Moreover, we shall show below that

$$(5.5) \quad D(A) = \{u \in V : A(u) \in L^2(G) \text{ and } \partial_A u = 0\} ,$$

where ∂_A is the abstract boundary operator given by

$$(5.4.b) \quad \partial_A u = \sum_{j=1}^n a(\partial_j u) \nu_j + ca(\gamma u)$$

on smooth functions $u \in D(A)$. Here $\vec{\nu} = (\nu_1, \dots, \nu_n)$ is the unit outward normal on ∂G . It follows, then, that the value of Au is determined by the *formal operator* (5.4.a), the restriction of $\mathcal{A}(u)$ to $C_0^\infty(G)$, and the domain $D(A)$ is determined by both the stable boundary conditions imposed by $u \in V$ and the variational boundary conditions $\partial_A u = 0$ obtained from the abstract Green's formula as the extension of (5.4.b).

We shall use the abstract Green's theorem to verify (5.5) and, hence, that the domain of A consists of those vectors which satisfy both the stable boundary conditions specified by V and the variational or complementary boundary conditions specified by ∂_A . First we recall the Green's formula from II.5. Let V, B be $\mathcal{D}$ Banach spaces and $\gamma : V \rightarrow B$ a strict homomorphism with kernel V_0 . Then γ is an abstract *trace operator* and its dual γ^* is an isomorphism of B' onto the annihilator $V_0^\perp$ in V' . Let H be a Hilbert space for which we identify $H = H'$, let $V \subset H$ with a continuous injection, and let V_0 be dense in H . Thus we have $H \subset V'$ and $H \subset V_0'$ by restriction. Assume $\mathcal{A} : V \rightarrow V'$ is given and define the corresponding *formal operator* $Au = \mathcal{A}u|_{V_0}$ as the indicated restriction. Set $D = \{u \in V : Au \in H\}$; this is the domain of the abstract *boundary operator*. For each $u \in D$, we find $\mathcal{A}u - Au \in V_0^\perp$, so there is a unique $\partial_A u \in B'$ for which $\mathcal{A}u - Au = \gamma^*(\partial_A u)$ in V' . This defines $\partial_A : D \rightarrow B'$, and we have

$$(5.6) \quad \mathcal{A}u(v) = (Au, v)_H + \partial_A u(\gamma v), \quad u \in D, v \in V.$$

This is the *abstract Green's formula* for $\mathcal{A}$.

Next we use (5.6) to characterize the restriction of $\mathcal{A}$ to H . Recall that we defined $D(A) = \{u \in V : Au \in H\}$. If $u \in D(A)$, then there is an $f \in H$ for which $Au = f$ in V' , hence, by (5.5)

$$(Au, v)_H + \partial_A u(\gamma v) = (f, v)_H, \quad v \in V.$$

This holds for each $v \in V_0$, and V_0 is dense in H , so we obtain successively $Au = f$ in H and $\partial_A u = 0$ in B' . Conversely, if $u \in D$ and $\partial_A u = 0$, it follows from (5.5) that $u \in D(A)$ and $Au = f$ in H . This completes the proof of (5.5).

We illustrate this further with the examples above. V is a subspace of $W^{1,p}(G)$ and $\mathcal{A} : V \rightarrow V'$ is given by (5.3). Let γ be the trace operator from V into $L^p(\partial G)$ and let B be the range of γ (restricted to V) with the inherited quotient norm. Then the kernel of γ is $V_0 = W_0^{1,p}(G)$, and this contains $C_0^\infty(G)$ and so is dense in $H = L^2(G)$. Note also that $B \hookrightarrow L^p(\partial G)$ and $L^{p'}(\partial G) \subset B'$. The formal part of $\mathcal{A}$ is computed from (5.3), and it is given by (5.4). The boundary operator is computed on the sufficiently smooth functions $u \in D$ to be given by

$$\partial_A u(\psi) = \int_G \left(\sum_{j=1}^n a(\partial_j u) \nu_j + c(s) a(u(s)) \right) \psi(s) ds, \quad \psi \in Rg(\gamma).$$

This follows directly from (5.3), (5.4) and (5.6). When we apply Proposition 5.1 in this situation we obtain a generalized solution u of the nonlinear parabolic equation

$$(5.7.a) \quad \partial_t u - \sum_{j=1}^n \partial_j a(\partial_j u) + ba(u) = f, \quad x \in G, t > 0,$$

and this solution satisfies $u(t) \in D(A)$ at each $t \geq 0$. If $V = W_0^{1,p}(G)$, then $\partial_A = 0$ so the last condition in (5.5) is vacuous and the boundary conditions are obtained

solely from $u(t) \in V$; that is,

$$(5.7.b) \quad u(s, t) = 0, \quad \text{a.e. } s \in \partial G, \quad t > 0.$$

If $V = W^{1,p}(G)$, then the boundary conditions are obtained solely from $\partial_A u(t) = 0$, so we have

$$(5.7.c) \quad \sum_{j=1}^n a(\partial_j u(s, t)) \nu_j(s) + c(s) a(u(s, t)) = 0, \quad \text{a.e. } s \in \partial G, \quad t > 0.$$

Thus, we can solve (5.7.a) subject to the boundary condition (5.7.b) of *Dirichlet type*, and we can likewise solve (5.7.a) subject to the *Neumann type* boundary condition (5.7.c). Of course, one needs also to specify the appropriate initial data $u_0 \in D(A)$ in either problem in order to obtain a well-posed problem. Other types of boundary conditions, for example, of mixed or non-local type, can be achieved by a corresponding choice of V . For each such choice, $W_0^{1,p} \subset V \subset W^{1,p}$, there will be a *pair* of boundary constraints implicit in $D(A)$. See II.5 for some examples.

The preceding examples can be easily modified so as to obtain first-order derivatives in the formal operator (5.4). Such examples show that in some sense the Proposition 5.1 is sharp: the Cauchy problem can be *reversible*, hence, there exists a solution *only if* the initial data u_0 belongs to $D(A)$. (See I.5.) By contrast, those Cauchy problems associated with a derivative (see Proposition III.4.2), or more generally with a subgradient, are well-posed for any u_0 in the space H , hence, they are *non-reversible*. This suggests we consider the extension of the notion of regular accretive operators to those which are multi-valued.

Assume the spaces V and H are given as above and let $\mathcal{A}$ be a monotone relation from V to V' , i.e., $\mathcal{A} \subset V \times V'$. Then we define a relation A on H by $A = \mathcal{A} \cap (V \times H)$, so its domain is $D(A) = \{u \in V : \mathcal{A}(u) \cap H \neq \emptyset\}$. Such an operator A is accretive on H . An important example is the case that arises from a subdifferential.

LEMMA 5.2. *Let V and H be given as above, and let $\varphi : V \rightarrow \mathbb{R}_\infty$ be proper, convex and lower-semi-continuous. Extend φ to all of H as $\varphi(v) = +\infty$ if $v \in H$, $v \notin V$. Assume there is a $q > 1$ such that*

$$(5.8) \quad \lim_{\|v\| \rightarrow \infty} \left(|v|_H^q + \varphi(v) \right) = +\infty.$$

Then $\text{dom } \varphi \subset V$ and φ is lower-semi-continuous on H .

PROOF. The first conclusion is trivial and the second follows easily. If $v_n \rightarrow v$ in H and $\varphi(v_n) \rightarrow a$, then (5.8) shows some subsequence of $\{v_n\}$ converges weakly in V to v . This leads directly to a proof of the second claim. $\square$

The Lemma 5.2 gives easy sufficient conditions for the hypotheses in the following result.

PROPOSITION 5.2. *Let the spaces V and H be given as above, and let $\varphi : H \rightarrow \mathbb{R}_\infty$ be convex and lower-semi-continuous with $\text{dom } \varphi \cap V \neq \emptyset$.*

(a) *If $\partial_H \varphi$ is the subgradient and $\partial \varphi$ is the subdifferential of the restriction $\varphi : V \rightarrow \mathbb{R}_\infty$, then*

$$\partial_H \varphi \cap (V \times H) \subset \partial \varphi \cap (V \times H).$$

- (b) If $\text{dom}(\partial_H \varphi) \subset V$, then $\partial_H \varphi = \partial \varphi \cap (V \times H)$. Thus, for each $f \in L^2(0, T; H)$ and $u_0 \in \overline{\text{dom}(\varphi)}$, there exists a unique $u \in C([0, T], H)$ which satisfies $u \in W^{1,2}(\delta, T; H)$ for $0 < \delta < T$, $u(t) \in D(A)$ at a.e. $t \in (0, T)$, $\sqrt{t} \frac{du}{dt} \in L^2(0, T; H)$, $\varphi(u(\cdot)) \in L^1(0, T)$, $u(0) = u_0$ and

$$(5.9) \quad \frac{du}{dt}(t) + \partial \varphi(u(t)) \ni f(t), \quad \text{a.e. } t \in (0, T),$$

Also, if $u_0 \in \text{dom}(\varphi)$ then $u \in W^{1,2}(0, T; H)$.

PROOF.

- (a) If $f \in H$, $u \in V$ and $f \in \partial_H \varphi(u)$, then

$$(f, v - u)_H \leq \varphi(v) - \varphi(u), \quad v \in H$$

and so it follows, from $f \in V'$ and by applying this to $v \in V$, that $f \in \partial \varphi(u)$.

- (b) In this case, $\partial_H \varphi = \partial_H \varphi \cap (V \times H)$ is maximal accretive and $\partial \varphi \cap (V \times H)$ is accretive in H , so by part (a) they are equal. The last part follows from Theorem 4.3. $\square$

The case (b) holds in the situation of Lemma 5.2, and then the subgradient $A = \partial_H \varphi$ is exactly the accretive operator on H obtained from the subdifferential $\mathcal{A} = \partial \varphi$. In the case where $\mathcal{A}$ is a G -differential, we have already obtained such results as Proposition III.4.2. For examples of parabolic initial-boundary-value problems which are resolved by Proposition 5.2, we have (5.7) with initial data any u_0 in $L^2(G)$. The details follow easily, as was described at the end of III.4. Moreover, Proposition 5.2 applies to problems with a multi-valued operator A , such as those described in II.8 and in IV.5. These Cauchy problems are *strictly parabolic* in the sense that initial data u_0 in the large space $\text{dom}(\varphi)$, the closure in H of $\text{dom}(\varphi)$, leads to a solution u whose value $u(t)$ at any $t > 0$ belongs to the smaller set $D(A)$ of smoother functions which satisfy the boundary conditions. This is a *regularizing property* of the parabolic problem.

We turn now to evolution problems associated with the stationary problem of Theorem II.4 or Proposition 2.2, that is, for equations of the form

$$(5.10) \quad \frac{du}{dt}(t) + \mathcal{A}(u(t)) + \partial \varphi(u(t)) \ni f(t), \quad \text{a.e. } t \in (0, T),$$

where $\mathcal{A}$ is a function from V to V' and φ is a convex function on V . This can be regarded as a *perturbation* of either (5.2) or (5.9). First we record the conditions sufficient to give a solution of (5.10) with $f(t) - u'(t) \in H$ at a.e. $t \in (0, T)$. Then we show that this solution satisfies $u(t) \in D(A)$, i.e., $\mathcal{A}(u(t)) \in H$, under hypotheses of *compatibility* of $\mathcal{A}$ and φ . This is an abstract *regularity* result which has strong implications for various semilinear parabolic equations and for related variational inequalities. The point here is to separate the operator in (5.10) into a smoothing or regularizing part and a singular part which does not destroy the smoothing effect of the other.

LEMMA 5.3. *Let V and H be given as above, let $\mathcal{A} : V \rightarrow V'$ be bounded, monotone and demicontinuous, let $\varphi : V \rightarrow \mathbb{R}_\infty$ be convex, lower-semi-continuous, assume $\varphi(0) = 0$ and*

$$(5.11) \quad \lim_{\|v\| \rightarrow \infty} \frac{|v|_H^2 + \mathcal{A}v(v) + \varphi(v)}{|v|_H} = +\infty.$$

Then $(\mathcal{A} + \partial\varphi) \cap (V \times H)$ is m -accretive on H .

PROOF. The restriction of $\mathcal{A} + \partial\varphi$ to H is clearly accretive, so it suffices to show that for each $f \in H$ there is a

$$u \in V : u + \mathcal{A}u + \partial\varphi(u) \ni f .$$

For this it suffices by Theorem II.4 that there be a $R > 0$ such that $\|v\| > R$ imply $\langle v + \mathcal{A}v - f, v \rangle + \varphi(v) > 0$. But $f \in H$, so this follows from (5.11). $\square$

PROPOSITION 5.3. *Let the spaces V and H be given as above, and let $\varphi : V \rightarrow \mathbb{R}_\infty$ be convex and lower-semi-continuous with $\varphi(0) = 0$. Assume $\mathcal{A} : V \rightarrow V'$ is bounded, monotone and demicontinuous.*

- (a) *If (5.11) holds, then for each $u_0 \in V$ with $(\mathcal{A}u_0 + \partial\varphi(u_0)) \cap H \neq \emptyset$ and each $f \in W^{1,1}(0, T; H)$ there exists a unique $u \in W^{1,\infty}(0, T; H)$ with $u(0) = u_0$ and (5.10), and we also have*

$$f(t) - D^+u(t) \in (\mathcal{A}(u(t)) + \partial\varphi(u(t))) \cap H , \quad t \in [0, T] .$$

- (b) *If A is m -accretive and if*

$$(5.12) \quad 0 \leq \varphi((I + \varepsilon A)^{-1}v) \leq \varphi(v) , \quad v \in V , \varepsilon > 0 ,$$

then $u(t) \in D(A)$ for $0 \leq t \leq T$, hence,

$$f(t) - \frac{du}{dt}(t) - A(u(t)) \in \partial\varphi(u(t)) \cap H , \quad \text{a.e. } t \in (0, T) .$$

PROOF. Part (a) follows from Theorem 4.1, since Lemma 5.3 shows $(\mathcal{A} + \partial\varphi) \cap V \times H$ is m -accretive on H . To prove (b), it suffices to show that if there exists a $g \in (\mathcal{A}(u) + \partial\varphi(u)) \cap H$, then $u \in D(A)$. Thus, let $g - \mathcal{A}(u) \in \partial\varphi(u)$ with $g \in H$, i.e.,

$$u \in V : \langle g - \mathcal{A}u, v - u \rangle \leq \varphi(v) - \varphi(u) , \quad v \in V .$$

For each $\varepsilon > 0$, there is a unique $u_\varepsilon = (I + \varepsilon A)^{-1}u$, and we set $v = u_\varepsilon$ in the above to obtain

$$\langle g - \mathcal{A}u, u_\varepsilon - u \rangle \leq 0 , \quad \varepsilon > 0 .$$

Since $\mathcal{A}$ is monotone, this implies

$$\langle g - \mathcal{A}u_\varepsilon, u_\varepsilon - u \rangle = (g - A(u_\varepsilon), -\varepsilon A(u_\varepsilon))_H \leq 0 .$$

This shows $|A(u_\varepsilon)|_H \leq |g|_H$, $\varepsilon > 0$, so (by passing to a subsequence and changing notation)

$$|u_\varepsilon - u|_H \leq \varepsilon |g|_H , \quad A(u_\varepsilon) \rightharpoonup h \text{ in } H .$$

Since A is m -accretive, Proposition 1.6 shows that $h = A(u)$, hence, $u \in D(A)$ as desired. $\square$

REMARKS. By taking the case of $\varphi \equiv 0$ we see that Proposition 5.3 contains Proposition 5.1. However it does not contain Proposition 5.2 since it does not imply the regularizing property and in fact does apply to some reversible evolution equations. The estimate (5.12) is the *compatibility condition* between $\mathcal{A}$ and $\partial\varphi$; see Lemma 5.4 below for alternative statements of it and compare (2.6).

We shall develop two examples to illustrate the application of Proposition 5.3 to initial-boundary-value problems for nonlinear parabolic partial differential equations. These examples contain variational inequalities as special cases, and it will be clear from the constructions that much more general problems could be obtained similarly. Also see II.8 and IV.2 above.

EXAMPLE 5.A. Assume G is a bounded domain in $\mathbb{R}^n$, ∂G is a C^1 manifold with unit outward normal ν , $2 \leq p < \infty$, and $\varphi_0 : \mathbb{R} \rightarrow \mathbb{R}_\infty$ is non-negative, convex, lower-semi-continuous, with $\varphi_0(0) = 0$. Set $V = W^{1,p}(G)$ and define

$$\varphi(v) = \frac{1}{p} \int_G \sum_{j=1}^n |\partial_j v(x)|^p dx + \int_{\partial G} \varphi_0(\gamma v(s)) ds, \quad v \in V.$$

The principle part of φ is continuous so as in II.8 we can compute the subdifferential $\partial\varphi$ termwise by Proposition II.7.7. The formal operator, obtained by restricting $\partial\varphi(u)$ to $C_0^\infty(G)$, is

$$B(u) = - \sum_{j=1}^n \partial_j a(\partial_j u), \quad u \in V,$$

where $a(\xi) = |\xi|^{p-1} \operatorname{sgn}(\xi)$. If $Bu \in H = L^2(G)$, we obtain the boundary condition

$$\partial_B(u) + \partial\varphi_0(\gamma u) \ni 0 \quad \text{in } L^{p'}(\partial G)$$

from the abstract Green's Theorem, where ∂_B is the boundary operator given by

$$\partial_B u = \sum_{j=1}^n a(\partial_j u) \nu_j$$

on smooth functions, and $\partial\varphi_0$ is the realization of $\partial\varphi_0 \subset \mathbb{R} \times \mathbb{R}$ in $L^{p'}(\partial G)$.

Let $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ be a monotone, continuous function with $\alpha(0) = 0$ and

$$|\alpha(\xi)| \leq C(|\xi|^{p-1} + 1), \quad \xi \in \mathbb{R}.$$

The operator $\mathcal{A} : L^p(G) \rightarrow L^{p'}(G)$ defined by

$$\mathcal{A}(u)(x) = \alpha(u(x)), \quad \text{a.e. } x \in G,$$

is bounded, continuous and monotone; see II.3 for such substitution or Nemitskyi operators. It follows that $\mathcal{A}$, the restriction to H , is m -accretive on H ; see Proposition II.8.1. From Proposition II.5 follows the estimate

$$\left(\sum_{j=1}^n \|\partial_j v\|_{L^p(G)}^p \right)^{1/p} + \|v\|_{L^2(G)} \geq c_0 \|v\|_{W^{1,p}(G)}, \quad v \in W^{1,p}(G);$$

this shows that (5.11) holds. Moreover, $(I + \varepsilon A)^{-1}(v) = (I + \varepsilon \alpha)^{-1}(v)$, and $(I + \varepsilon \alpha)^{-1}$ is Lipschitz and monotone, so (5.12) is true and Proposition 3(b) holds. Assume $u_0 \in V$ satisfies

$$B(u_0) \in L^2(G), \quad -\partial_B(u_0) \in \partial\varphi_0(\gamma u_0), \quad \alpha(u_0) \in L^2(G),$$

and that $f \in W^{1,1}(0, T; L^2(G))$ is given. Then there exists a unique

$$u \in W^{1,\infty}(0, T; L^2(G))$$

which is a generalized solution of the initial-boundary-value problem

$$(5.13.a) \quad \frac{\partial u}{\partial t}(x, t) - \sum_{j=1}^n \partial_j a(\partial_j u(x, t)) + \alpha(u(x, t)) = f(x, t), \quad \text{a.e. } x \in G, t > 0,$$

$$(5.13.b) \quad \sum_{j=1}^n a(\partial_j u) \nu_j(s) + \partial\varphi_0(u(s, t)) \ni 0, \quad \text{a.e. } s \in \partial G, t > 0,$$

$$(5.13.c) \quad u(x, 0) = u_0(x), \quad \text{a.e. } x \in G.$$

The parabolic equation (5.13.a) and the boundary condition (5.13.b) follow from (5.10); the regularizing effect is that $\alpha(u(\cdot, t)) \in L^2(G)$ for each $t > 0$. Hence, each of the four terms in (5.13(a)) individually belongs to $L^2(G)$ at each $t > 0$, so (5.13.b) has both terms in $L^{p'}(\partial G) \subset B'$, and it can be interpreted as above by the abstract Green's Theorem.

In Example 5.A it is the higher-order operator $\partial\varphi$ through which the additional regularity is most effective. The operator $\mathcal{A}$ is more regular and thereby makes it easier to verify the compatibility condition (5.12). Much more general Dirichlet integrands could be used here, as well as other useful and interesting convex functions such as those in II.8. Moreover, we remark that the restrictions on the data, u_0 and f , can be relaxed by using Proposition 5.2. This follows since $\mathcal{A} + \partial\varphi$ is actually a subdifferential in Example 5.A; thus, it suffices for *existence* of a solution to require that

$$u_0 \in L^2(G), \quad f \in L^2(G \times (0, T)).$$

The first condition follows from the observation that $C_0^\infty(G) \subset \text{dom}(\varphi) \cap D(A)$ and $C_0^\infty(G)$ is dense in $L^2(G)$. Therefore (5.13) is a *quasilinear parabolic problem* with the *regularizing property*: at a.e. $\delta > 0$, $u(\delta)$ is in the domain of $(\mathcal{A} + \partial\varphi) \cap V \times H$ and then Proposition 5.3(b) applies. For this we need only require additionally that $f \in W^{1,1}(\delta, T; L^2(G))$ for every $0 < \delta < T$.

REMARKS.

- (1) In the special situation of (5.13) a solution of (5.10) satisfies $\alpha(u(\cdot, t)) \in L^{p'}(G)$ for each $t > 0$, even without the regularizing effect obtained from Proposition 5.3(b). Thus we have $B(u(t)) \in L^{p'}(G)$; this is sufficient to construct the boundary operator $\partial_B(u(t)) \in B'$. This follows from the discussion of II.5 of the Green's Theorem; we merely choose $\|\cdot\|_{L^p(G)}$ as the continuous seminorm on V , and thus $L^{p'}(G)$ is obtained as the pivot space which determines the domain of the boundary operator.
- (2) Variational inequalities are included in Example 5.A. To illustrate this, consider the convex function

$$(5.14) \quad \varphi_0(x) = \begin{cases} 0, & x \geq 0, \\ +\infty, & x < 0. \end{cases}$$

Its subgradient is given by

$$r \in \partial\varphi_0(s) \iff s \geq 0, \quad r \leq 0, \quad rs = 0.$$

Thus the boundary condition (5.13.b) is equivalent to

$$\gamma(u) \geq 0, \quad \partial_B(u) \geq 0, \quad \partial_B u(\gamma u) = 0,$$

and this has a “pointwise a.e.” characterization since $B \subset L^p(\partial G)$. This can be regarded as a *boundary control* problem in which one seeks to show there is a non-negative flux, $\partial_B u$, which will maintain a non-negative boundary temperature, γu , and for which no heat flux is supplied at those points on the boundary where the temperature is positive. By taking similar functions in the Dirichlet integrand defining φ , one could also obtain such variational inequalities with constraints on the gradient, ∇u .

Before considering the next example, we give a useful characterization of the *compatibility condition* (5.12) in the special case where φ is given on H .

PROPOSITION 5.4 (BREZIS-PAZY). *Let A be an m -accretive function on H and $\varphi : H \rightarrow \mathbb{R}_\infty$ be proper, convex and lower-semi-continuous. The following are equivalent:*

- (i) $\varphi((I + \varepsilon A)^{-1}x) \leq \varphi(x)$, $x \in H$, $\varepsilon > 0$.
- (ii) $(A_\varepsilon x, w) \geq 0$, $w \in \partial\varphi(x)$, $\varepsilon > 0$.
- (iii) $(A_\varepsilon x, \varphi'_\alpha(x)) \geq 0$, $x \in H$, $\varepsilon > 0$, $\alpha > 0$.
- (iv) $(Ax, \varphi'_\alpha(x)) \geq 0$, $x \in D(A)$, $\alpha > 0$.
- (v) $\varphi_\alpha((I + \varepsilon A)^{-1}x) \leq \varphi_\alpha(x)$, $x \in H$, $\alpha > 0$.

PROOF. Assume (i). If $w \in \partial\varphi(x)$ then

$$-\varepsilon(A_\varepsilon x, w) = ((I + \varepsilon A)^{-1}x - x, w) \leq \varphi((I + \varepsilon A)^{-1}x) - \varphi(x) \leq 0$$

so (ii) holds. Assume (ii). Since A_ε is monotone

$$(A_\varepsilon x - A_\varepsilon((I + \alpha\partial\varphi)^{-1}x), \quad x - ((I + \alpha\partial\varphi)^{-1}x)) \geq 0,$$

and $x - (I + \alpha\partial\varphi)^{-1}x = \alpha\varphi'_\alpha(x)$, we obtain

$$(A_\varepsilon x, \varphi'_\alpha(x)) \geq (A_\varepsilon((I + \alpha\partial\varphi)^{-1}x), \varphi'_\alpha(x)) \geq 0$$

from $\varphi'_\alpha(x) \in \partial\varphi((I + \alpha\partial\varphi)^{-1}x)$ and (ii). Letting $\varepsilon \rightarrow 0$ shows that (iii) implies (iv). For any $x \in H$ we have for $y = (I + \varepsilon A)^{-1}x$

$$\begin{aligned} \varphi_\alpha(x) - \varphi_\alpha(y) &\geq (\varphi'_\alpha(y), x - y) = (\varphi'_\alpha(y), \varepsilon A_\varepsilon y) \\ &= \varepsilon(\varphi'_\alpha(y), Ay) \geq 0 \end{aligned}$$

by (iv), so (iv) implies (v). Finally, note that (i) follows by letting $\alpha \rightarrow 0$ in (v). $\square$

We have already seen examples of such estimates in Example 2.F and Example 2.G, and in Proposition II.9.3 where A is a linear elliptic operator in divergence form, and φ is a convex integrand on $L^p(G)$. This can be extended to nonlinear A as we illustrate with the following simple case.

EXAMPLE 5.B. Assume that the domain G , the functions $a(\cdot)$, φ_0 , and the spaces $V = W^{1,p}(G)$, $H = L^2(G)$ are just as given in Example 5.A. Assume $b \in L^\infty(G)$ and $c \in L^\infty(\partial G)$ are both non-negative, and define $\mathcal{A} : V \rightarrow V'$ by (5.3). Furthermore, define the convex integrand

$$\varphi(v) = \int_G \varphi_0(v(x)) \, dx \quad \text{if } \varphi_0(v) \in L^1(G), \quad +\infty \text{ otherwise,}$$

as in Proposition II.8.1. To check that (5.12) is satisfied, it suffices by Proposition 5.4(iv) to verify that

$$(Au, \sigma(u))_{L^2(G)} \geq 0, \quad u \in D(A)$$

for any monotone Lipschitz function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ with $\sigma(0) = 0$. Recall that the Yosida approximation $\sigma = (\varphi_0)_\alpha$ has such properties; see Proposition 1.9. But for $u \in D(A) \subset V$ we have $v = \sigma(u) \in V$, and in (5.3) we obtain

$$\begin{aligned} (Au, \sigma(u))_{L^2(G)} &= \int_G \left\{ \sum_{j=1}^n (\partial_j u)^p \sigma'(u(x)) + b(x)(u(x))^p \right\} dx \\ &\quad + \int_{\partial G} c(s)(\gamma u(s))^p \, ds \geq 0. \end{aligned}$$

The estimate (5.1) holds, so A is m -accretive by Lemma 5.1 and we have the situation of Proposition 5.3(b). Assume $u_0 \in V$ satisfies

$$A(u_0) \in L^2(G), \quad \partial_A(u_0) = 0, \quad u_0 \in \text{dom}(\partial \varphi_0),$$

and that $f \in W^{1,1}(0, T; L^2(G))$ is given. Then there exists a unique generalized solution $u \in W^{1,\infty}(0, T; L^2(G))$ of the initial-boundary-value problem

$$(5.15.a) \quad \frac{\partial u}{\partial t}(x, t) - \sum_{j=1}^n \partial_j a(\partial_j u(x, t)) + b(x)a(u(x)) + v(x, t) = f(x, t),$$

$$(5.15.b) \quad v(x, t) \in \partial \varphi_0(u(x, t)), \quad \text{a.e. } x \in G, t > 0,$$

$$(5.15.c) \quad \sum_{j=1}^n a(\partial_j u) \nu_j(s) + c(s)a(\gamma u(s)) = 0, \quad \text{a.e. } s \in \partial G, t > 0,$$

$$(5.15.d) \quad u(x, 0) = x_0(x), \quad \text{a.e. } x \in G.$$

From the fact that the solution of (5.10) satisfies $u(t) \in D(A)$, with $D(A)$ given by (5.5), we obtain the boundary condition (5.15.c) from (5.4.b). This also implies that (5.15.b) holds with $v(\cdot, t) \in L^2(G)$ for each $t \geq 0$. (From Proposition 5.3(a) we could only have deduced that $v(t) \in V'$, and then $\partial_A(u(t))$ and (5.15.c) would be meaningless.) Finally we note that the operator $\mathcal{A}$ above is actually a (sub) differential, so we can apply Proposition 5.2 and obtain a solution from more general data, u_0, f .

For an example of a *variational inequality* we choose φ_0 as in (5.14). Then (5.15.b) is equivalent to

$$u(x, t) \geq 0, v(x, t) \leq 0, u(x, t)v(x, t) = 0, \quad \text{a.e. } x \in G, t > 0.$$

As before this can be regarded as a distributed control problem in which a non-negative source, $-v(x, t)$, will maintain a non-negative temperature, $u(x, t)$, and the source is active (non-zero) only where the temperature is zero.

It is clear that there can be many different ways to obtain a specific initial-boundary-value problem in the abstract form (5.10). For example, each of (5.13) and (5.15) can be modified slightly to contain the other if $\partial\varphi_0$ is a function of the form of α . In Example 5.B the regularity, $u(t) \in D(A)$, is effective on both operators: on the higher-order $\mathcal{A}$ because it gives meaning to (5.15.c), and on $\partial\varphi$ because (5.15.b) contains functions with a pointwise meaning. The compatibility condition (5.12) is easy to verify because of the simple pointwise form of φ . It is clear that more general operators $\mathcal{A}$ could have been used. For example, with appropriate boundary conditions one could introduce first-order derivatives in $\mathcal{A}$; see Proposition II.9.3 for an example of linear $\mathcal{A}$ of second-order elliptic type with first-order derivatives.

IV.6. Semilinear Degenerate Evolution Equations

We have previously considered implicit evolution equations, i.e., those in which the solution is not necessarily differentiated, or even differentiable, but rather some function of the solution is differentiated in the equation. Such equations were resolved directly in III.3 and III.6. Here we give an algebraic construction which directly extends the preceding results to a rather large and very useful class of such equations.

Let E be a real vector space and denote its algebraic dual space by E^* . Let $\mathcal{B} : E \rightarrow E^*$ be linear, symmetric and non-negative, i.e., *monotone*. This determines a semi-scalar-product

$$b(x, y) = \mathcal{B}x(y), \quad x, y \in E,$$

and we denote the corresponding seminorm space by E_b . Its continuous dual E'_b is a Hilbert space. Finally, let $A \subset E \times E'_b$ be a relation with domain $D = \{x \in E : A(x) \neq \varnothing\}$; this will be regarded as a multi-valued operator as before, i.e., $[x, w] \in A$ if and only if $w \in A(x)$. A *solution* of the semilinear equation in E'_b

$$(6.1) \quad \frac{d}{dt}(\mathcal{B}u(t)) + A(u(t)) \ni f(t), \quad 0 < t < T,$$

is a function $u : [0, T] \rightarrow E$ for which $\mathcal{B}u \in C([0, T], E'_b)$, $\mathcal{B}u(\cdot)$ is absolutely continuous on each $[\delta, T]$, $0 < \delta < T$, hence, differentiable a.e., and (6.1) holds a.e. on $(0, T)$. The *Cauchy problem* for (6.1) is to find a solution u for which $(\mathcal{B}u)(0) = w_0$ is specified in E'_b . We shall show that (6.1) is equivalent to the standard evolution equation (4.1) and, from this correspondence, determine conditions on $\mathcal{B}$, A , f and w_0 for which the Cauchy problem is well-posed.

For the moment let's consider the special case in which $b(\cdot, \cdot)$ is a scalar-product and E_b is complete, i.e., E_b is a Hilbert space. Then $\mathcal{B}$ is the Riesz isomorphism onto E'_b . Furthermore if A is monotone then any two solutions u_1, u_2 of (6.1) with respective data f_1, f_2 satisfy

$$\frac{d}{dt} \frac{1}{2} \|u_1(t) - u_2(t)\|_b^2 \leq \|f_1(t) - f_2(t)\|_{E'_b} \|u_1(t) - u_2(t)\|_b, \quad 0 \leq t \leq T,$$

and thus Lemma 4.1 implies that

$$\|u_1(t) - u_2(t)\|_b \leq \|u_1(0) - u_2(0)\|_b + \int_0^t \|f_1(s) - f_2(s)\|_{E'_b} ds, \quad 0 \leq t \leq T.$$

This estimate shows that when A is monotone E_b is the appropriate space in which solutions are estimated, hence, the space in which to seek solutions. Also, the solution of the Cauchy problem should be represented by a semigroup which is generated by some realization of $-\mathcal{B}^{-1}A$. These observations will be developed below. Note that if we identify $E_b \cong E'_b$ then $\mathcal{B}$ is the identity and (6.1) is the standard evolution equation studied previously.

In this special case where E_b is Hilbert space, (6.1) is equivalent to the equation in E_b

$$\frac{du}{dt} + \mathcal{B}^{-1} \circ A(u(t)) \ni \mathcal{B}^{-1}f(t), \quad 0 < t < T,$$

so we are led to ask whether the composite operator $\mathcal{A} \equiv \mathcal{B}^{-1} \circ A$ is m -accretive on E_b . Here we define $[x, y] \in \mathcal{A}$ if and only if $\mathcal{B}y = w$ for some $[x, w] \in A$, and then we have

$$(y, z)_{E_b} = b(y, z) = \mathcal{B}y(z) = \langle w, z \rangle, \quad z \in E.$$

From here it follows that $\mathcal{A}$ is accretive in E_b if A is monotone in $E \times E'_b$. Also the range condition, $Rg(I + \mathcal{A}) = E_b$, is fulfilled if and only if $Rg(\mathcal{B} + A) = E'_b$. Thus from Theorem 4.1 it follows that for each absolutely continuous $f : [0, T] \rightarrow E'_b$ and each $u_0 \in D$ there is a unique absolutely continuous solution $u : [0, T] \rightarrow E_b$ of (6.1) with $u(0) = u_0$.

We give two examples to illustrate this situation where $\mathcal{B}$ is a Riesz isomorphism.

EXAMPLE 6.A PSEUDOPARABOLIC EQUATION. Let $V = W_0^{1,p}(G)$, $p \geq 2$, and suppose the given operator $A : V \rightarrow V'$ is monotone and continuous. Let $b(\cdot) \in L^\infty(G)$ with $b(x) \geq 0$, a.e. $x \in G$, and define

$$\langle \mathcal{B}u, v \rangle = \int_G (u(x)v(x) + b(x)\nabla u(x) \cdot \nabla v(x)) dx, \quad u, v \in V.$$

The completion of V under the continuous seminorm $\langle \mathcal{B}\cdot, \cdot \rangle^{\frac{1}{2}}$ defines the space E_b , and the imbeddings $H_0^1(G) \hookrightarrow E_b \hookrightarrow L^p(G)$ are continuous. It follows that $V \hookrightarrow E_b$ is continuous and dense and that $E'_b \subset (H_0^1)' = H^{-1}$. We will have $Rg(\mathcal{B} + A) \supset E'_b$ if $\mathcal{B} + A : V \rightarrow V'$ is *onto*, and this is the case here since $\mathcal{B} + A$ is type M, bounded and coercive. It follows from Theorem 4.1 that the Cauchy problem for (6.1) has a unique solution for each $u_0 \in E_b$ with $A(u_0) \in E'_B$, i.e., for each $u_0 \in \text{dom}(\mathcal{A})$.

The operator $\mathcal{B} : V \rightarrow \mathcal{D}(G)^*$ is given by $\mathcal{B}u = u - \nabla \cdot b(\cdot) \nabla u$. A solution $u(t)$ of the Cauchy problem for (6.1) corresponds to a solution of the the mixed *parabolic-pseudoparabolic* partial differential equation with initial and first type boundary conditions

$$u(t) \in W_0^{1,p}(G) : \frac{\partial}{\partial t}(u(t) - \nabla \cdot b(\cdot) \nabla u(t)) + A(u(t)) = f(t), \quad t \in (0, T),$$

$$u(x, 0) = u_0(x).$$

Here the initial condition $u_0(\cdot)$ is given as above, and the equation holds in $L^\infty(0, T; E'_b)$. The operator A can be chosen as in Example II.5.A, and corresponding problems with boundary conditions of other types can as easily be treated. See Example III.6.A. Note that the equation here holds in the much smaller space E'_b , i.e., $A(u(\cdot)) \in L^\infty(0, T; E'_b)$, and this is a *regularity* result for the solution. The condition on the initial data above will be relaxed in Theorem 6.1 below when A is a subdifferential.

EXAMPLE 6.B POROUS MEDIUM EQUATION. We saw in Example III.6.C how to choose the spaces and operators to realize the porous medium equation III.6.10. We repeat the construction here in order to use Theorem 4.1 to obtain the H^{-1} -solution in a more general setting. Let G be a bounded domain in $\mathbb{R}^n$ and assume $\frac{2n}{n+2} \leq p < \infty$. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}_\infty$ be proper, convex, lower semi-continuous and define the convex integrand $\Phi : L^p(G) \rightarrow \mathbb{R}_\infty$, $1 \leq p < \infty$, by

$$\Phi(u) = \int_G \varphi(u(x)) \, dx \quad \text{if } \varphi(u) \in L^1(G), \quad +\infty \text{ otherwise.}$$

According to Proposition II.8.1, the function Φ is proper, convex, lower semi-continuous, and its subdifferential is given by $f \in \partial\Phi(u)$ if and only if

$$f \in L^{p'}(G), u \in L^p(G) \text{ and } f(x) \in \partial\varphi(u(x)), \quad \text{a.e. } x \in G.$$

Define $V = L^p(G)$ and set

$$A(u) = \partial\Phi(u), \quad u \in L^p(G).$$

The Riesz map $\mathcal{R} : H_0^1 \rightarrow H^{-1}$ is the isomorphism defined by the scalar product

$$\mathcal{R}\varphi(\psi) = (\varphi, \psi)_{H_0^1} \equiv \int_G \nabla\varphi \cdot \nabla\psi \, dx,$$

so we have $\mathcal{R} = -\Delta$. The corresponding scalar product on H^{-1} is given by

$$(f, g)_{H^{-1}} \equiv (\mathcal{R}^{-1}f, \mathcal{R}^{-1}g)_{H_0^1}, \quad f, g \in H^{-1},$$

and it satisfies

$$(f, g)_{H^{-1}} = \langle f, \mathcal{R}^{-1}g \rangle = \langle \mathcal{R}^{-1}f, g \rangle, \quad f, g \in H^{-1}.$$

Define

$$\mathcal{B}u(v) = (u, v)_{H^{-1}}, \quad u, v \in H^{-1}.$$

The completion of V under the continuous seminorm $\langle \mathcal{B}\cdot, \cdot \rangle^{\frac{1}{2}}$ determines the space $E_b = H^{-1}$. We have the continuous imbedding $V \hookrightarrow E_b$, i.e., $E'_b = H_0^1(G) \hookrightarrow L^{p'}(G) = V'$ by the Sobolev imbedding Theorem II.4.3 since $p' \leq 2n/(n-2)$.

The multi-valued operator A is monotone, so the composition $\mathcal{A} = \mathcal{B}^{-1} \cdot A$ is accretive, and it remains to show that $Rg(\mathcal{B} + A) = E_b$. That is, for each $f \in E_b$ we want to solve

$$u \in H^{-1} : \quad \mathcal{B}u + A(u) \ni f \text{ in } H_0^1,$$

and this is equivalent to solving

$$w \in H_0^1 : \quad A^{-1}(w) - \Delta w \ni -\Delta f \text{ in } H^{-1}.$$

Now if the monotone graph $A^{-1}(\cdot)$ is a *function* which satisfies the growth condition

$$|A^{-1}(\xi)| \leq K(1 + |\xi|^{p'-1}), \quad \xi \in \mathbb{R},$$

then the methods of Section II.5 suffice to show that $Rg(A^{-1}(\cdot) - \Delta) = H^{-1}$. More generally, if the convex function φ satisfies a lower bound

$$\varphi(\xi) \geq K\left(\frac{|\xi|^p}{p} - 1\right), \quad \xi \in \mathbb{R},$$

then the *conjugate* convex function φ^* , whose subgradient is A^{-1} , satisfies the dual upper bound

$$\varphi^*(\xi) \leq K^*\left(\frac{|\xi|^{p'}}{p'} + 1\right), \quad \xi \in \mathbb{R}.$$

The corresponding convex integrand on $L^{p'}(G)$ is then continuous, and the methods of Section II.8 suffice. Note that either condition implies that $A(\cdot)$ is necessarily *onto*.

We shall assume that $F \in W^{1,\infty}(0, T; H^{-1}(G))$ and define

$$f \in W^{1,\infty}(0, T; H_0^1(G))$$

by

$$f(t)(v) = \int_G ((-\Delta)^{-1}F)(x, t)v(x) dx, \quad v \in V.$$

Assume that $u_0 \in L^p(G)$ and there is a $w \in A(u_0) \cap H_0^1(G)$. From the preceding discussion and Theorem 4.1 we obtain a solution $u : [0, T] \rightarrow L^p(G)$ of the *porous medium equation*

$$\begin{aligned} \frac{\partial u(t)}{\partial t} - \Delta w(t) &= F(t) \text{ in } H^{-1}(G), \quad w(t) \in \partial\Phi(u(t)) \text{ for a.e. } t \in (0, T), \\ w &\in L^\infty(0, T; H_0^1(G)), \quad \lim_{t \rightarrow 0} u(t) = u_0 \text{ in } H^{-1}(G), \end{aligned}$$

with each term of the evolution equation in $L^\infty(0, T; H^{-1})$. This procedure requires more of the initial function than does Example III.6.C, but it yields a somewhat stronger solution.

Next, let $\mathcal{B}$ be as originally given above, i.e., linear, symmetric and monotone on the vector space E . From the Cauchy-Schwartz inequality

$$|b(x, y)|^2 \leq b(x, x)b(y, y), \quad x, y \in E$$

it follows that $\mathcal{B}$ is continuous from E_b into E'_b . Denote the kernel of $\mathcal{B}$ by

$$K = \{x \in E : \mathcal{B}x = 0\} = \{x \in E : b(x, x) = 0\}$$

and the corresponding quotient space by E/K . Thus, each element of E/K is a coset, $\tilde{x} = x + K = \{x + y : y \in K\}$, and we can define a scalar-product on this space by

$$(6.2) \quad \tilde{b}(\tilde{x}, \tilde{y}) = b(x, y), \quad x, y \in E.$$

The completion of E/K with the corresponding norm is a Hilbert space W whose scalar product is an extension of $\tilde{b}(\cdot, \cdot)$. Let q denote the canonical quotient map, $q(x) = \tilde{x}, x \in E$. Then q is a strict homomorphism of E_b into W , with a dense range, and so its dual map $q' : W' \rightarrow E'_b$, given by

$$q'(g)(x) = g(q(x)), \quad g \in W', \quad x \in E_b,$$

is an isomorphism. Let $\mathcal{B}_0 : W \rightarrow W'$ be the Riesz isomorphism given by

$$\mathcal{B}_0 v(w) = (v, w)_W, \quad v, w \in W.$$

For each pair $x, y \in E_b$ we have by (6.2)

$$\mathcal{B}x(y) = \mathcal{B}_0(q(x))(q(y)) = q'\mathcal{B}_0q(x)(y),$$

so we have factored the given operator into the form

$$(6.3) \quad \mathcal{B} = q'\mathcal{B}_0q.$$

This will play a fundamental role below.

To obtain a similar factorization of the (possibly multi-valued) operator $A : D \rightarrow E'_b$ we define $A_0 \subset W \times W'$ by

$$\begin{aligned} g \in A_0(\tilde{x}) \text{ if and only if there exists} \\ \text{an } x \in D \text{ with } q(x) = \tilde{x} \text{ and } q'(g) \in A(x). \end{aligned}$$

Thus, we obtain a relation A_0 with domain $q[D] = \{\tilde{x} : x \in D\}$ for which

$$(6.4) \quad A = q'A_0q.$$

Note that q is not one-to-one, so A_0 need not be single-valued, even if A is a function! The connections between these operators are given as follows.

LEMMA 6.1.

- (a) For each $\tilde{x} = q(x) \in W$, $q'(g) \in A(x)$ if and only if $g \in A_0(\tilde{x})$, and in this case, we have $q'(g)(x) = g(\tilde{x})$.
- (b) The inclusion $q'(g) \in (\mathcal{B} + A)(x)$ holds if and only if $g \in (\mathcal{B}_0 + A_0)(\tilde{x})$.

COROLLARY 6.1.

- (a) $A : D \rightarrow E'_b$ is monotone if and only if $A_0 : q[D] \rightarrow W'$ is monotone.
- (b) q' is a bijection of $Rg(\mathcal{B}_0 + A_0)$ onto $Rg(\mathcal{B} + A)$.

Let the operators $\mathcal{B}$ and A be given as in Lemma 6.1 and suppose that u is a solution of (6.1). Since q' is an isomorphism of W' onto E'_b , it follows from (6.3) and (6.4) that $\tilde{u} \equiv q \cdot u$ is a solution of

$$(6.5) \quad \frac{d}{dt}(\mathcal{B}_0\tilde{u}(t)) + A_0(\tilde{u}(t)) \ni \tilde{f}(t), \quad 0 < t < T,$$

where $\tilde{f}(t) = (q')^{-1}f(t)$. Conversely, if $\tilde{u}$ is a solution of (6.5), then for a.e. $t \in (0, T)$, $\tilde{u}(t) \in q[D]$, so there is a $u(t) \in D$ with $q(u(t)) = \tilde{u}(t)$. Since q' is an isomorphism and $\mathcal{B}u(t) = q'\mathcal{B}_0\tilde{u}(t)$, $0 < t < T$, it follows that u is a solution of (6.1). This proves the following.

LEMMA 6.2. A function $u : [0, T] \rightarrow E$ is a solution of (6.1) if and only if $\tilde{u} = q \cdot u : [0, T] \rightarrow W$ is a solution of (6.5).

From Theorem 4.1 and the correspondence between the operators in (6.1) and (6.5) and between the solutions of these equations as given in Lemma 6.1 and Lemma 6.2, we obtain the first two parts of our main result.

THEOREM 6.1. *Let the linear, symmetric and monotone operator $\mathcal{B}$ be given from the real vector space E to its algebraic dual E^* , and let E'_b be the Hilbert space which is the dual of E with the seminorm*

$$|x|_b = \mathcal{B}x(x)^{1/2}, \quad x \in E.$$

Let $A \subset E \times E'_b$ be a relation with domain $D = \{x \in E : A(x) \neq \varnothing\}$.

(a) *Assume A is monotone. If u_j is a solution of*

$$\frac{d}{dt}(\mathcal{B}u_j(t)) + A(u_j(t)) \ni f_j(t), \quad 0 < t < T,$$

for $j = 1, 2$, then it follows that

$$(6.6) \quad |u_1(t) - u_2(t)|_b \leq |u_1(0) - u_2(0)|_b + \int_0^t \|f_1(s) - f_2(s)\|_{E'_b} ds, \quad 0 \leq t \leq T.$$

If $f_1 = f_2$ and if $\mathcal{B}u_1(0) = \mathcal{B}u_2(0)$ then $\mathcal{B}u_1(t) = \mathcal{B}u_2(t)$ for all $0 \leq t \leq T$. Furthermore, if $\mathcal{B} + A$ is strictly monotone then there is at most one solution of the Cauchy problem for (6.1).

(b) *Assume A is monotone and $Rg(\mathcal{B} + A) = E'_b$. Then, for each $u_0 \in D$ and each $f \in W^{1,1}(0, T; E'_b)$, there is a solution u of (6.1) with*

$$\mathcal{B}u \in W^{1,\infty}(0, T; E'_b), \quad u(t) \in D, \quad \text{all } 0 \leq t \leq T, \quad \text{and } \mathcal{B}u(0) = \mathcal{B}u_0.$$

(c) *Let A be the subdifferential, $\partial\varphi$, of a convex lower-semi-continuous function $\varphi : E_b \rightarrow [0, +\infty]$ with $\varphi(0) = 0$. Then for each u_0 in the E_b -closure of $\text{dom}(\varphi)$ and each $f \in L^2(0, T; E'_b)$ there is a solution u of (6.1) with*

$$\varphi \circ u \in L^1(0, T), \quad \sqrt{t} \frac{d}{dt} \mathcal{B}u(\cdot) \in L^2(0, T; E'_b), \quad u(t) \in D, \quad \text{a.e. } t \in [0, T],$$

and $\mathcal{B}u(0) = \mathcal{B}u_0$. If in addition $u_0 \in \text{dom}(\varphi)$ then

$$\varphi \circ u \in L^\infty(0, T), \quad \frac{d}{dt} \mathcal{B}u \in L^2(0, T; E'_b).$$

PROOF. In view of the preceding development it suffices to show that part (c) follows from Theorem 4.3. Now φ is lower-semi-continuous on E_b so it follows that φ is constant on each coset $\tilde{x} \in E/K$. This means that a function $\tilde{\varphi} : W \rightarrow [0, \infty]$ is defined implicitly by $\varphi = \tilde{\varphi} \circ q$ and the effective domain of $\tilde{\varphi}$ is $\text{dom}(\tilde{\varphi}) = q[\text{dom}(\varphi)]$. It is a direct consequence of the definitions that

$$(6.7) \quad \partial\varphi = q' \circ \partial\tilde{\varphi} \circ q.$$

Specifically, $f \in \partial\varphi(x)$ if and only if $f = q'(g)$ with $g \in \partial\tilde{\varphi}(\tilde{x})$. Thus, comparing (6.4) and (6.7) we find $A_0 = \partial\tilde{\varphi}$. The proof now follows from the preceding case. $\square$

Next we describe how the situation of Theorem 1 can be attained by operators of the type determined by nonlinear elliptic boundary-value problems. The following is typical.

DEFINITION. Assume X is a reflexive Banach space, $\mathcal{A} : X \rightarrow X'$ is monotone and demicontinuous, $\mathcal{B} : X \rightarrow X'$ is continuous, linear, symmetric and monotone. Then $\mathcal{B}, \mathcal{A}$ is a *regular pair* on X .

For such a pair it follows from Theorem II.2.2 that if $f \in X'$ and there is an $R > 0$ such that

$$\mathcal{B}v(v) + \mathcal{A}v(v) > f(v)$$

for all v with $\|v\| > R$, then $f \in Rg(\mathcal{B} + \mathcal{A})$. This proves the following result, which yields the situation of Theorem 1(b).

LEMMA 6.3. *If $\mathcal{B}, \mathcal{A}$ is a regular pair on X , and if*

$$(6.8) \quad \lim_{\|v\|_X \rightarrow \infty} \frac{\mathcal{B}v(v) + \mathcal{A}v(v)}{\mathcal{B}v(v)^{1/2}} = +\infty,$$

then $Rg(\mathcal{B} + \mathcal{A}) \supset E'_b$, where E_b is the space X with the seminorm $|v|_b = \mathcal{B}v(v)^{1/2}$, $v \in X$.

To obtain the situation of Theorem 1(b), we define $D = \{v \in X : \mathcal{A}(v) \in E'_b\}$ and $A(x) = \mathcal{A}(x)$ for $x \in D$.

PROPOSITION 6.1. *Assume $\mathcal{B}, \mathcal{A}$ is a regular pair on X and (6.8) holds. Then for each $u_0 \in D$ and $f \in W^{1,1}(0, T; E'_b)$ there exists a solution u of (6.1) with $\mathcal{B}u \in W^{1,\infty}(0, T; E'_b)$, $u(t) \in D$ for all $t \in [0, T]$, and $\mathcal{B}u(0) = \mathcal{B}u_0$.*

A typical application of Proposition 6.1 would be elliptic-parabolic initial-boundary-value problems of the form III.6.9 where $\mathcal{B}$ corresponds to multiplication by a function $b(x) \geq 0$. When $X = W^{1,p}(G)$, such a $\mathcal{B}$ is continuous if $b \in L^{q^*}(G)$, where $q^* = p^*/(p^* - 2)$ and $W^{1,p}(G) \hookrightarrow L^{p^*}(G)$ is continuous.

The boundedness of $\mathcal{B}$ can be relaxed as follows.

LEMMA 6.4. *Assume V is a reflexive Banach space, $\mathcal{A} : V \rightarrow V'$ is monotone and demicontinuous; $E_b = \{E, b(\cdot, \cdot)\}$ is a semi-scalar-product space, and $\mathcal{B} : E \rightarrow E'_b$ is the canonical operator, $\mathcal{B}u(v) = b(u, v)$; and $E \cap V$ is dense in V and dense in E_b . Set $X = E \cap V$, $\|v\|_X = \|v\|_V + b(v, v)^{1/2}$, $v \in X$. If X is complete, then $\mathcal{B}, \mathcal{A}$ is a regular pair on X .*

PROOF. Since the embeddings $V' \hookrightarrow X'$, $E'_b \hookrightarrow X'$, $X \hookrightarrow V$, $X \hookrightarrow E_b$ are all continuous, it suffices to show X is reflexive. Consider the map $x \mapsto [x, \tilde{x}] : X \rightarrow V \times W$, where W is the completion of E_b/K as in the proof of Theorem 6.1. This is an isomorphism of X onto a closed subspace of a product of reflexive spaces, so X is reflexive. $\square$

COROLLARY 6.2. *If E_b is complete then $\mathcal{B}, \mathcal{A}$ is a regular pair on X .*

PROOF. Let X_0 be the completion of X . Then the imbeddings $X_0 \hookrightarrow V$ and $X_0 \hookrightarrow E_b$ are continuous, so $X_0 \subset V \cap E_b = X$ and X is complete. $\square$

We consider a related class of *second-order evolution equations*. Let V_a be a semi-scalar-product space whose semi-scalar-product is given by the linear symmetric $\mathcal{A} : V_a \rightarrow V'_a$ and whose seminorm is $|x|_a = \mathcal{A}x(x)^{1/2}$. Similarly let $\mathcal{C} : V_c \rightarrow V'_c$ be the linear symmetric operator determined by the semi-scalar-product on V_c ,

and denote the seminorm by $|x|_c = \mathcal{C}x(x)$. Set $W = V_a \cap V_c$ with seminorm $|x|_W \equiv (\mathcal{A} + \mathcal{C})x(x)^{1/2}$. We shall assume

W is dense in V_a and in V_c

so we can identify $V'_a \hookrightarrow W'$ and $V'_c \hookrightarrow W'$ by restriction. Suppose we are given a relation B from W to W' , i.e., $B \subset W \times W'$, and a pair of functions $f_a : [0, T] \rightarrow V'_a, f_c : [0, T] \rightarrow V'_c$. Then we consider the evolution system

$$(6.9.a) \quad \frac{d}{dt} \mathcal{A}u(t) - \mathcal{A}v(t) = f_a(t), \quad 0 < t < T,$$

$$(6.9.b) \quad \frac{d}{dt} \mathcal{C}v(t) + \mathcal{A}u(t) + B(v(t)) \ni f_c(t).$$

A *solution* of (6.9) is a pair of functions $u : [0, T] \rightarrow V_a, v : [0, T] \rightarrow W$ such that each of $\mathcal{A}u : [0, T] \rightarrow V'_a$ and $\mathcal{C}v : [0, T] \rightarrow V'_c$ are continuous on $[0, T]$, absolutely continuous on $[\delta, T]$ for each $0 < \delta < T$, and (6.9) holds at a.e. $t \in (0, T)$. Note that necessarily $v(t) \in \text{dom}(B)$ and that $\mathcal{A}u(t) + B(v(t))$ contains an element of V'_c at each such $t \in (0, T)$. The *Cauchy problem* for (6.9) is to find a solution for which

$$\mathcal{A}u(0) = w_1 \in V'_a, \quad \mathcal{C}v(0) = w_2 \in V'_c$$

where w_1 and w_2 are given. We shall obtain the following result as a direct consequence of Theorem 6.1.

THEOREM 6.2. *Let the spaces V_a, V_c, W , the linear, symmetric, monotone operators $\mathcal{A}, \mathcal{C}$, and the relation B be given as above.*

(a) *Assume B is monotone. If u^j, v^j are solutions of*

$$(6.10.a) \quad \frac{d}{dt} \mathcal{A}u^j(t) - \mathcal{A}v^j(t) = f_a^j(t)$$

$$(6.10.b) \quad \frac{d}{dt} \mathcal{C}v^j(t) + \mathcal{A}u^j(t) + B(v^j(t)) \ni f_c^j(t), \quad 0 < t < T$$

for $j = 1, 2$, then it follows that

$$(6.11) \quad (|u^1(t) - u^2(t)|_a^2 + |v^1(t) - v^2(t)|_c^2)^{1/2} \\ \leq (|u^1(0) - u^2(0)|_a^2 + |v^1(0) - v^2(0)|_c^2)^{1/2} \\ + \int_0^t (|f_a^1(s) - f_a^2(s)|_{V'_a}^2 + |f_c^1(s) - f_c^2(s)|_{V'_c}^2)^{1/2} ds, \quad 0 \leq t \leq T.$$

If u^j, v^j are solutions of (6.9) for $j = 1, 2$ and if $\mathcal{A}u^1(0) = \mathcal{A}u^2(0), \mathcal{C}v^1(0) = \mathcal{C}v^2(0)$, then for a.e. $t \in [0, T]$ we have

$$(6.12) \quad \mathcal{A}u^1(t) = \mathcal{A}u^2(t), \quad \mathcal{A}v^1(t) = \mathcal{A}v^2(t), \quad \mathcal{C}v^1(t) = \mathcal{C}v^2(t)$$

and the selections $w^j(t) = f_c(t) - (\mathcal{C}v^j(t))' - \mathcal{A}u^j(t) \in B(v^j(t))$ satisfy $\langle w^1(t) - w^2(t), v^1(t) - v^2(t) \rangle = 0$. Thus, if $\mathcal{A} + B + \mathcal{C}$ is strictly-monotone, then $v^1(t) = v^2(t)$, and if $\mathcal{A}$ is strictly-monotone, then also $u^1(t) = u^2(t)$.

(b) *Assume B is monotone, that the seminormed space V_a is complete, and $Rg(\mathcal{A} + B + \mathcal{C}) = W'$. Then for each pair $w_1 \in V'_a, u_2 \in W$ with $(w_1 + B(u_2)) \cap V'_c$ non-empty, and each pair $f_a : [0, T] \rightarrow V'_a, f_c : [0, T] \rightarrow V'_c$ of absolutely-continuous functions, there exists a solution of (6.9) with*

$$\mathcal{A}u(0) = w_1, \quad \mathcal{C}v(0) = \mathcal{C}u_2.$$

PROOF. In order to obtain Theorem 6.2 as a consequence of Theorem 6.1, set $E = V_a \times V_c$, the indicated product space, and define $\mathcal{B}\vec{u} = [\mathcal{A}u, \mathcal{C}v] \in V'_a \times V'_c = E'_b$ for $\vec{u} = [u, v] \in E$. Next define the relation $A \subset E \times E'_b$ by $\vec{g} \in A(\vec{u})$ if for some $w \in B(v)$, $\vec{g} = [-\mathcal{A}v, \mathcal{A}u + w] \in V'_a \times V'_c$. Thus the domain of A is

$$D = \{\vec{u} \in V_a \times V_a : \mathcal{A}u + B(v)) \cap V'_c \text{ is non-empty} \} .$$

In case (a) it is clear that A is monotone so (6.11) follows from (6.6). The first and last equalities in (6.12) follow from (6.11), and the second follows from (6.10.a). This finishes the proof of (a).

To prove (b) we need to verify the range condition on $\mathcal{B} + A$. For this purpose, note the equivalence of the equation

$$\vec{u} \in D : (\mathcal{B} + A)\vec{u} \ni \vec{f} , \quad \vec{f} \in E'_b$$

and the system

$$\begin{aligned} v \in W : (\mathcal{A} + B + \mathcal{C})v &= f_c - f_a \in W' , \\ u \in V_a : \mathcal{A}u &= \mathcal{A}v + f_a \in V'_a . \end{aligned}$$

The first has a solution by the range condition, and the second is solvable because V_a is complete, hence, $Rg(\mathcal{A}) = V'_a$. Note that any such solution u, v satisfies $\mathcal{A}u + B(v) \ni f_c - \mathcal{C}v \in V'_c$, so $[u, v] \in D$. Thus (b) is obtained from the corresponding part of Theorem 6.1.

The system (6.9) is actually equivalent to a single equation of second-order. By eliminating the first component we obtain the equation

$$(6.13) \quad \frac{d}{dt} \left(\frac{d}{dt} \mathcal{C}v(t) + Bv(t) - f_c(t) \right) + \mathcal{A}v(t) = f_a(t) , \quad 0 < t < T .$$

A *solution* of (6.13) is a function $v : [0, T] \rightarrow V_c$ such that $\mathcal{C}v : [0, T] \rightarrow V'_c$ is absolutely continuous with $v(t) \in D(B)$ at a.e. $t \in [0, T]$, $\frac{d}{dt} \mathcal{C}v + B(v) - f_c : [0, T] \rightarrow V'_a$ is absolutely continuous and (6.13) holds at a.e. $t \in (0, T)$. The *Cauchy problem* for (6.13) is to find a solution for which

$$\mathcal{C}v(0) = v_1 \in V'_c , \quad \left(\frac{d}{dt} \mathcal{C}v + B(v) - f_c \right) \Big|_{t=0} = v_2 \in V'_a$$

are specified. It is clear that the Cauchy problems for (6.9) and for (6.13) are equivalent, so Theorem 6.2 gives the following.

COROLLARY 6.3. *Let the spaces V_a, V_c, W , the linear operator $\mathcal{A}, \mathcal{C}$ and the relation B be given as above.*

- (a) *If B is monotone and $\mathcal{A} + B + \mathcal{C}$ is strictly monotone, then there is at most one solution of the Cauchy problem for (6.13).*
- (b) *If B is monotone, V_a is complete, and $Rg(\mathcal{A} + B + \mathcal{C}) = W'$, then for each pair $w_0 \in W$, $v_0 \in V'_a$ with $(v_0 + B(w_0)) \cap V'_c$ non-empty, and each pair $f_a : [0, T] \rightarrow V'_a$, $f_c : [0, T] \rightarrow V'_c$ of absolutely continuous functions, there exists a solution of (6.13) with*

$$\mathcal{C}v(0) = \mathcal{C}v_0 , \quad \left(\frac{d}{dt} (\mathcal{C}v) + B(v) - f_c \right) (0) = v_o .$$

REMARKS. If v_1 and v_2 are solutions of (6.13) corresponding to two pairs of data

$$w_0^j, v_0^j, \quad f_a^j, f_c^j, \quad j = 1, 2,$$

then an estimate like (6.11) holds for

$$|v^1(t) - v^2(t)|_c^2 + \left| \frac{d}{dt} C(v^1(t) - v^2(t)) + B^0 v^1(t) - B^0 v^2(t) + (f_c^2(t) - f_c^1(t)) \right|_{V'_a}^2$$

where $B^0 v^j(t)$ denotes the selection from $B(v^j(t))$ determined by the solution v^j .

Theorem 6.1 is obtained as the special case of Corollary 6.3 or Theorem 6.2 that results from the choice of $\mathcal{A} = 0$. Thus the three results are equivalent!

In the special case where $\mathcal{A}$ is an isomorphism, i.e., the Riesz map of the Hilbert space V_a onto its dual, we obtain an equivalent second-order equation which characterizes (essentially) the first component u in (6.9). This is a smoother notion of solution and requires correspondingly smoother data for existence. A *solution* of

$$(6.14) \quad \frac{d}{dt} \left(C \left(\frac{dw}{dt} \right) \right) + B \left(\frac{dw}{dt} \right) + \mathcal{A}w(t) = f_a(t) + f_c(t), \quad 0 < t < T$$

is an absolutely continuous $w : [0, T] \rightarrow V_a$ such that $\frac{dw}{dt}(t) \in V_c$ at a.e. $t \in [0, T]$, $C(\frac{dw}{dt}) : [0, T] \rightarrow V'_c$ is absolutely continuous and (6.14) holds at a.e. $t \in (0, T)$. The Cauchy problem for (6.14) is to find a solution of (6.14) which satisfies

$$w(0) = w_0, \quad C \left(\frac{dw}{dt} \right)(0) = Cw_1$$

where $w_0 \in V_a$ and $w_1 \in V_c$ are given. From the change-of-variable

$$w(t) = u(t) - \mathcal{A}^{-1} f_a(t), \quad f_a(t) = \frac{d}{dt} f_a(t)$$

we obtain the following.

COROLLARY 6.4. *Let the spaces V_a, V_c, W , the linear operators $\mathcal{A}, C$ and relation B be given as above.*

- (a) *If B is monotone and $\mathcal{A} + B + C$ is strictly-monotone, then there is at most one solution of (6.14).*
- (b) *If V_a is Hilbert space, so $\mathcal{A}$ is isomorphism, B is monotone, and $Rg(\mathcal{A} + B + C) = W'$ then for each pair $w_0 \in V_a, w_1 \in D(B)$ with $(\mathcal{A}w_0 + B(w_1) + f_a(0)) \cap V'_c$ non-empty and each pair $f_a[0, T] : \rightarrow V'_a$ with $\frac{d}{dt} f_a$ absolutely continuous and $f_c : [0, T] \rightarrow V'_c$ absolutely continuous, there exists a solution of the Cauchy problem for (6.14).*

IV.7. Accretive Operators in Banach Space

Here we begin by extending the notion of accretive operators from Hilbert space to Banach space. Many of the properties of such operators hold in this more general setting, and we list some of them. Then we briefly indicate how the proofs of existence and uniqueness given previously in Hilbert space for the evolution equation carry over to those special Banach spaces for which the duality map plays the role of the scalar product, namely, those Banach spaces whose dual is uniformly convex. The well-posedness of the Cauchy problem in a general Banach space is a much more difficult topic, and we shall develop it in Section 8.

Let X be a real Banach space. For any non-empty subset C of X we define $|C| \equiv \inf\{\|x\| : x \in C\}$. A *relation* or graph on X is a subset of $X \times X$. If A is a relation on X , we define its domain $D(A) = \{x : [x, y] \in A\}$, range $Rg(A) = \{y : [x, y] \in A\}$ and inverse $A^{-1} = \{[y, x] : [x, y] \in A\}$. Such a relation can be regarded as a function into the subsets of X with $A(x) = \{y : [x, y] \in A\}$, and then A is a (graph of a) function exactly when $A(x)$ is single-valued. We define the usual linear combinations by

$$\begin{aligned}\lambda A &= \{[x, \lambda y] : [x, y] \in A\} \\ A + B &= \{[x, y + z] : [x, y] \in A \text{ and } [x, z] \in B\}.\end{aligned}$$

DEFINITION. The relation A on X is *accretive* if $[x_j, y_j] \in A$ for $j = 1, 2$ implies

$$\|x_1 - x_2\| \leq \|(x_1 + \alpha y_1) - (x_2 + \alpha y_2)\|$$

for all $\alpha > 0$.

This is clearly equivalent to requiring that $J_\alpha \equiv (I + \alpha A)^{-1}$ be a contraction on $Rg(I + \alpha A)$ for each $\alpha > 0$. We shall also define $A_\alpha \equiv \frac{1}{\alpha}(I - J_\alpha)$ on $Rg(I + \alpha A)$.

PROPOSITION 7.1. Let A be accretive and $\alpha > 0$.

- (a) A_α is Lipschitz with constant $\frac{2}{\alpha}$ and accretive on $Rg(I + \alpha A)$.
- (b) $A_\alpha(x) \in A \circ J_\alpha(x)$ for $x \in Rg(I + \alpha A)$.
- (c) $\|A_\alpha(x)\| \leq |A(x)|$, $x \in D(A) \cap Rg(I + \alpha A)$.
- (d) $\lim_{\alpha \rightarrow 0^+} J_\alpha(x) = x$, $x \in \overline{D(A)} \cap \bigcap_{\alpha > 0} Rg(I + \alpha A)$

PROOF.

- (a) For $j = 1, 2$ let $y_j = (I + \beta A_\alpha)(x_j)$. Then $y_j = (I + \frac{\beta}{\alpha}(I - J_\alpha))(x_j)$ so

$$y_1 - y_2 + \frac{\beta}{\alpha}(J_\alpha x_1 - J_\alpha x_2) = \left(1 + \frac{\beta}{\alpha}\right)(x_1 - x_2),$$

hence,

$$\left(1 + \frac{\beta}{\alpha}\right)\|x_1 - x_2\| \leq \|y_1 - y_2\| + \frac{\beta}{\alpha}\|J_\alpha x_1 - J_\alpha x_2\|,$$

and we need only note that J_α is a contraction.

- (b) $A_\alpha x \in \frac{1}{\alpha}((I + \alpha A)J_\alpha x - J_\alpha x) = A(J_\alpha x)$.
- (c) $A_\alpha(x) = \frac{1}{\alpha}(J_\alpha(I + \alpha A)x - J_\alpha x) = \frac{1}{\alpha}(J_\alpha(x + \alpha y) - J_\alpha(x))$ for each $y \in Ax$, so $\|A_\alpha x\| \leq \|y\|$.
- (d) $\|J_\alpha x - x\| = \alpha\|A_\alpha x\| \leq \alpha|Ax|$ for $x \in D(A) \cap Rg(I + \alpha A)$. The convergence extends by uniform continuity to $x \in \overline{D(A)}$. $\square$

LEMMA 7.1. If A is accretive, then $Rg(I + \alpha A) = X$ for all $\alpha > 0$ if it holds for some $\alpha > 0$.

PROOF. If $Rg(I + \alpha A) = X$ for some $\alpha > 0$ and $\beta > \alpha/2$, then for $w \in X$ we have $(I + \beta A)x \ni w$, or equivalently, $(I + \alpha A)x \ni \frac{\alpha}{\beta}w + (1 - \frac{\alpha}{\beta})x$, and this is equivalent to $T(x) = x$ for the strict contraction

$$T(x) \equiv (I + \alpha A)^{-1} \left(\frac{\alpha}{\beta}w + \left(1 - \frac{\alpha}{\beta}\right)x \right), \quad x \in X.$$

Thus $Rg(I + \beta A) = X$ for all $\beta > \alpha/2$ and by induction for all $\beta > 0$. $\square$

We define the relation A to be *m-accretive* if it is accretive and the above range condition holds.

PROPOSITION 7.2. *Let A be m-accretive. Then (a) A is maximal accretive, (b) A is closed, and (c) if $x_\alpha \rightarrow x$ and $A_\alpha x_\alpha \rightarrow y$, then $[x, y] \in A$.*

PROOF.

(a) Suppose $[x_0, y_0] \in X \times X$ and

$$\|x_0 - x\| \leq \|(x_0 + \alpha y_0) - (x + \alpha y)\|, \quad \alpha > 0, [x, y] \in A.$$

Choose $[x, y] \in A$ such that $x + y = x_0 + y_0$. Then $x_0 = x$ and $y_0 = y$, so $[x_0, y_0] \in A$.

(b) Let $A \ni [x_n, y_n] \rightarrow [x_0, y_0]$ in $X \times X$. Since $\|x_n - x\| \leq \|(x_n + \alpha y_n) - (x + \alpha y)\|$, $\alpha > 0$, $[x, y] \in A$, we have $\|x_0 - x\| \leq \|(x_0 + \alpha y_0) - (x + \alpha y)\|$, $\alpha > 0$, $[x, y] \in A$. Part (a) then shows $[x_0, y_0] \in A$.

(c) Since $\{A_\alpha x_\alpha\}$ is bounded, $J_\alpha x_\alpha \rightarrow x$. But $A_\alpha x_\alpha \in A(J_\alpha x_\alpha)$, so the result follows by (b). $\square$

The accretiveness of the relation A on X can be determined by the normalized duality map $J : X \rightarrow X'$. Indeed, comparing Proposition II.8.4 with Proposition II.8.6, we find that A is accretive if and only if $[x_j, y_j] \in A$ for $j = 1, 2$ implies there is an $f \in J(x_1 - x_2)$ such that $f(y_1 - y_2) \geq 0$. By definition of J , this f is characterized by

$$f \in X' : f(x_1 - x_2) = \|f\|^2 = \|x_1 - x_2\|^2.$$

This characterization clearly extends the Hilbert space case where J is the identity with the identification $H \cong H'$, and it permits the following extension of Theorem 4.1 to the case of uniformly convex X' .

THEOREM 7.1 (KATO). *Let A be m-accretive on the Banach space X , and assume that X' is uniformly convex. Let $u_0 \in D(A)$, $f \in W^{1,1}(0, T; X)$, and $\omega \in \mathbb{R}$. Then there exists a unique function $u \in W^{1,\infty}(0, T; X)$ for which $u(0) = u_0$ and*

$$(7.1) \quad \frac{du}{dt}(t) + A(u(t)) \ni \omega u(t) + f(t), \quad \text{a.e. } t \in [0, T].$$

PROOF. Suppose u_1 and u_2 are both solutions of (7.1); then we have

$$\frac{d}{dt}(u_1(t) - u_2(t)) + Au_1(t) - Au_2(t) \ni \omega(u_1(t) - u_2(t)).$$

By Proposition II.8.5 the function $\varphi(t) \equiv \|u_1(t) - u_2(t)\|$ is differentiable a.e., and

$$\|u_1(t) - u_2(t)\| \cdot \frac{d}{dt}\|u_1(t) - u_2(t)\| = f(u_1'(t) - u_2'(t)), \quad f = J(u_1(t) - u_2(t)).$$

(See Lemma 7.2 below for a direct proof.) Thus applying f to the difference above yields

$$\|u_1(t) - u_2(t)\| \cdot \frac{d}{dt}\|u_1(t) - u_2(t)\| \leq \omega \|u_1(t) - u_2(t)\|^2,$$

and from here we get

$$\|u_1(t) - u_2(t)\| \leq e^{\omega t} \|u_1(0) - u_2(0)\|, \quad t \geq 0;$$

this implies *uniqueness*.

The following is a direct verification of the differentiation-of-norm rule used above.

LEMMA 7.2. *Let u be weakly-differentiable at s , i.e., $\frac{d}{dt}f(u(t))|_{t=s} = f(u'(s))$ for every $f \in X'$, and assume $t \mapsto \|u(t)\|$ is G -differentiable at $t = s$. Then*

$$\|u(s)\| \frac{d}{ds} \|u(s)\| = f(u'(s)) \text{ for each } f \in J(u(s)).$$

PROOF. For each $h > 0$ we have

$$f(u(s+h) - u(s)) \leq (\|u(s+h)\| - \|u(s)\|) \|f\|, \quad f \in J(u(s)).$$

Since u is weakly differentiable at s_1 we obtain

$$f(u'(s)) \leq \left(\frac{d}{ds} \|u(s)\| \right) \|u(s)\|;$$

the reverse inequality follows similarly. $\square$

We continue with the proof of *existence* of a solution of (7.1). Consider the *approximate equations*

$$(7.2) \quad \frac{du_\alpha}{dt}(t) + A_\alpha(u_\alpha(t)) = \omega u_\alpha(t) + f(t), \quad 0 \leq t \leq T$$

with $u_\alpha(0) = u_0$ for each $\alpha > 0$. Here A_α is prescribed by Proposition 7.1; specifically, each A_α is Lipschitz continuous on all of X , so (7.2) has a unique solution $u_\alpha \in C^1([0, T]; X)$ with the indicated initial value.

Let $\alpha, \beta > 0$ be given. From Lemma 7.2 we obtain

$$\frac{1}{2} \frac{d}{dt} \|u_\alpha(t) - u_\beta(t)\|^2 + J(u_\alpha(t) - u_\beta(t)) (A_\alpha u_\alpha(t) - A_\beta u_\beta(t)) = \omega \|u_\alpha(t) - u_\beta(t)\|^2,$$

and this gives

$$\|u_\alpha(t) - u_\beta(t)\|^2 \leq -2 \int_0^t e^{2\omega(t-s)} J(u_\alpha - u_\beta) (A_\alpha u_\alpha - A_\beta u_\beta) ds.$$

Since $A_\alpha u_\alpha \in A(J_\alpha u_\alpha)$ and A is accretive, we get

$$\begin{aligned} & \|u_\alpha(t) - u_\beta(t)\|^2 \\ & \leq -2 \int_0^t e^{2\omega(t-s)} (J(u_\alpha - u_\beta) - J(J_\alpha u_\alpha - J_\beta u_\beta)) (A_\alpha u_\alpha - A_\beta u_\beta) ds \end{aligned}$$

and, hence,

$$(7.3) \quad \begin{aligned} & \|u_\alpha(t) - u_\beta(t)\|^2 \\ & \leq 2 \int_0^t e^{2\omega(t-s)} \|J(u_\alpha - u_\beta) - J(J_\alpha u_\alpha - J_\beta u_\beta)\| \cdot \|A_\alpha u_\alpha - A_\beta u_\beta\| ds. \end{aligned}$$

On the other hand, by differentiating (7.2) we get

$$\frac{d^2}{dt^2}u_\alpha + \frac{d}{dt}A_\alpha u_\alpha = \omega \frac{d}{dt}u_\alpha + \frac{df}{dt}, \quad \text{a.e. on } [0, T].$$

The accretiveness of A yields $J(\frac{du_\alpha}{dt})(\frac{d}{dt}A_\alpha u_\alpha) \geq 0$, so applying $J(\frac{du_\alpha}{dt})$ to this equation leads to

$$\frac{1}{2} \frac{d}{dt} \|u'_\alpha(t)\|^2 \leq \omega \|u'_\alpha(t)\|^2 + \|u'_\alpha(t)\| \cdot \|f'(t)\|.$$

Applying Lemma 4.1 then gives

$$(7.4) \quad \left\| \frac{du_\alpha(t)}{dt} \right\| \leq e^{\omega t} \left\| \frac{du_\alpha(0)}{dt} \right\| + \int_0^t e^{\omega(t-s)} \|f'(s)\| ds.$$

From (7.2) at $t = 0$ and $\|A_\alpha(u_0)\| \leq |A(u_0)|$ it follows that for some $k > 0$, $\|A_\alpha u_\alpha(t)\| \leq k$ for $0 \leq t \leq T$. Thus

$$\|u_\alpha(t) - J_\alpha u_\alpha(t)\| = \alpha \|A_\alpha u_\alpha(t)\| \leq \alpha k,$$

so $\{u_\alpha(t) - J_\alpha u_\alpha(t)\}$ converges to zero uniformly on $[0, T]$. Since J is uniformly continuous on bounded sets, (7.3) then shows that $u(t) = \lim_{\alpha \rightarrow 0} u_\alpha(t)$ exists in $C(0, T; X)$. Also (7.4) shows that u is Lipschitz on $[0, T]$, hence, absolutely continuous.

It remains to show that u is a solution of (7.1). Let $[x, y] \in A$ and set $x_\alpha = x + \alpha y$, so $y = A_\alpha(x_\alpha)$. Since A_α is accretive it follows from (7.2) that

$$\|u_\alpha(t) - x_\alpha\|^2 \leq \|u_\alpha(t_0) - x_\alpha\|^2 + 2 \int_{t_0}^t J(u_\alpha(s) - x_\alpha)(\omega u_\alpha(s) + f(s) - y) ds.$$

By letting $\alpha \rightarrow 0$ we get

$$(7.5) \quad \|u(t) - x\|^2 - \|u(t_0) - x\|^2 \leq 2 \int_{t_0}^t J(u(s) - x)(\omega u(s) + f(s) - y) ds.$$

Note that for any pair $v, w \in X$ we have

$$J(v)(w - v) \leq \|w\| \|v\| - \|v\|^2 \leq \frac{1}{2} (\|w\|^2 - \|v\|^2),$$

so from (7.5) we obtain

$$J(u(t_0) - x) \left(\frac{u(t) - u(t_0)}{t - t_0} \right) \leq \frac{1}{t - t_0} \int_{t_0}^t J(u(s) - x)(\omega u(s) + f(s) - y) ds.$$

Now if t_0 is a point in $(0, T)$ at which $\frac{du}{dt}(t_0)$ exists, then by letting $t \rightarrow t_0$ we obtain

$$J(u(t_0) - x) \left(-\frac{du}{dt}(t_0) + \omega u(t_0) + f(t_0) - y \right) \geq 0.$$

Since this holds for any $[x, y] \in A$ and A is maximal accretive by Proposition 7.2.a, we have

$$-\frac{du}{dt}(t_0) + \omega u(t_0) + f(t_0) \in A(u(t_0)).$$

Since u is differentiable at a.e. $t_0 \in [0, T]$ we have (7.1). □

LEMMA 7.3. *If A is m -accretive and B is accretive and Lipschitz, then $A + B$ is m -accretive.*

PROOF. Since B is single-valued, the characterization of accretive by the normalized duality map shows that $A + B$ is accretive. To show $A + B$ is m -accretive, it suffices to solve

$$x + \alpha Bx + \alpha A(x) \ni y \text{ for } y \in X \text{ and some } \alpha > 0 .$$

We may choose α such that αB is a strict contraction. But a solution of this equation is characterized by $x = (I + \alpha A)^{-1}(y - \alpha B(x))$, the fixed point of a strict contraction, so there always exists such a solution. $\square$

COROLLARY 7.1. *Let X be a Banach space with uniformly convex dual, X' . Let A be a relation on X such that for some $\omega_1 \geq 0$, $A + \omega_1 I$ is m -accretive. Let B be a Lipschitz function: for some $\omega_2 > 0$*

$$\|B(x) - B(y)\| \leq \omega_2 \|x - y\| , \quad x, y \in X .$$

Then for each $u_0 \in D(A)$, $f \in W^{1,1}(0, T; X)$ and $\omega \in \mathbb{R}$ there exists a unique $u \in W^{1,\infty}(0, T; X)$ for which $u(0) = u_0$ and

$$(7.6) \quad \frac{du}{dt}(t) + A(u(t)) + B(u(t)) \ni \omega u(t) + f(t) , \quad \text{a.e. } t \in [0, T] .$$

PROOF. Add $(\omega_1 + \omega_2)u(t)$ to both sides of (7.6) and note that $B + \omega_2 I$ is accretive. $\square$

One can also resolve the *periodic problem* when the operator is strongly accretive.

PROPOSITION 7.3. *Let X be a Banach space with uniformly convex dual, X' . Let A be an m -accretive relation on X for which $A - \omega I$ is accretive for some $\omega > 0$. Then for each $f \in W^{1,1}(0, T; X)$, there is a unique solution $u \in W^{1,\infty}(0, T; X)$ of*

$$(7.7) \quad \begin{aligned} \frac{du}{dt}(t) + A(u(t)) &\ni f(t) , \quad \text{a.e. } t \in [0, T] , \\ u(0) &= u(T) . \end{aligned}$$

PROOF. Define $K : D(A) \rightarrow \overline{D(A)}$ by $K(u_0) = u(T)$ where u is the solution of (7.7) with $u(0) = u_0$ given by Theorem 7.1. Then K is a strict contraction with $\|K(u_1) - K(u_2)\| \leq e^{-\omega T} \|u_1 - u_2\|$, so it extends by continuity to all of $\overline{D(A)}$. Moreover, it has a unique fixed point $u_0 \in \overline{D(A)}$. Furthermore, there is a sequence $u_n \in W^{1,\infty}(0, T; X)$ of solutions of (7.7) for which $u_n \rightarrow u$ in $C(0, T; X)$ and $u(0) = u(T) = u_0$.

We shall show that u is a solution of (7.7). Since $A - \omega I$ is accretive, for each n we have for $h > 0$

$$\|u_n(t+h) - u_n(t)\| \leq e^{-\omega t} \|u_n(h) - u_n(0)\| + \int_0^t e^{-\omega(t-s)} \|f(s+h) - f(s)\| ds .$$

By taking the limit as $n \rightarrow \infty$ we get

$$(7.8) \quad \|u(t+h) - u(t)\| \leq e^{-\omega t} \|u(h) - u(0)\| + \int_0^t e^{-\omega(t-s)} \|f(x+h) - f(s)\| ds$$

for all $t \in [0, T]$ and $h > 0$ with $t+h \in [0, T]$. By extending f and u to $t \in (T, 2T]$ by $f(t-T)$ and $u(t-T)$, respectively, we obtain

$$\begin{aligned} \|u(T+h) - u(T)\| &\leq e^{-\omega T} \|u(h) - u(0)\| \\ &+ \int_0^{T-h} e^{-\omega(T-s)} \|f(s+h) - f(s)\| ds \\ &+ \int_{T-h}^T e^{-\omega(T-s)} \|f(s+h) - f(T) + f(0) - f(s)\| ds . \end{aligned}$$

due to the jump in variation of f at T . The periodicity of u then gives

$$\begin{aligned} (1 - e^{-\omega T}) \|u(h) - u(0)\| &\leq \int_0^T e^{-\omega(T-s)} \|f(s+h) - f(s)\| ds \\ &+ \|f(T) - f(0)\| \frac{1 - e^{-\omega h}}{\omega} , \end{aligned}$$

and using this in (7.8) yields

$$\begin{aligned} \|u(t+h) - u(t)\| &\leq e^{-\omega t} (1 - e^{-\omega T})^{-1} \left\{ \int_0^T e^{-\omega(T-s)} \|f(s+h) - f(s)\| ds \right. \\ &\quad \left. + \|f(T) - f(0)\| \frac{1 - e^{-\omega h}}{\omega} \right\} + \int_0^t e^{-\omega(t-s)} \|f'(s)\| ds . \end{aligned}$$

Thus u is absolutely continuous on $[0, T]$ with

$$\begin{aligned} \|u'(t)\| &\leq e^{-\omega t} (1 - e^{-\omega T})^{-1} \left\{ \int_0^T e^{-\omega(T-s)} \|f'(s)\| ds \right. \\ &\quad \left. + \|f(T) - f(0)\| \right\} + \int_0^t e^{-\omega(t-s)} \|f'(s)\| ds . \end{aligned}$$

This shows $u \in W^{1,\infty}(0, T; X)$, and the technique of the proof of Theorem 7.1 will show that it is a solution of (7.7). $\square$

We shall apply the preceding results to *semilinear evolution equations* of the form

$$\frac{du}{dt} + Au + \alpha(u) \ni f$$

in which A is a linear operator on $L^p(G)$ and α is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$. This is a direct application of the results in II.9 which were motivated by the example of the *elliptic* operator

$$(7.9) \quad A_1 u = - \sum_{i,j=1}^n \partial_j (a_{ij} \partial_i u) + \sum_{i=1}^n \partial_i (a_i u) + au$$

with Dirichlet boundary conditions.

In order to show that the operator “ $A + \alpha$ ” is m -accretive in $L^p(G)$, we need to consider the corresponding stationary problem

$$u + \varepsilon(Au + v) = f, v \in \alpha(u)$$

for each $\varepsilon > 0$. In order to apply Theorem II.9.5, we shall assume the following:

- (i) G is a bounded domain in $\mathbb{R}^n$ and $A : D(A) \rightarrow L^1(G)$ is linear, $D(A)$ is dense in $L^1(G)$, and $(I + \lambda A)^{-1}$ is a contraction on $L^1(G)$ for every $\lambda > 0$.
- (ii) For each $f \in L^1(G)$ and $\lambda > 0$,

$$\sup_G (I + \lambda A)^{-1} f \leq (\sup_G f)^+.$$

- (iii) α is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \alpha(0)$.

In this situation it follows by Theorem II.9.5 that for each $\varepsilon > 0$ and $f \in L^1(G)$ there is a unique pair $u \in D(A)$, $v \in L^1(G)$ such that

$$(7.10) \quad u + \varepsilon(Au + v) = f, \quad v(x) \in \alpha(u(x)), \quad \text{a.e. } x \in G.$$

Moreover we can apply Proposition II.9.3 to obtain estimates on any such solution. If $f \in L^p(G)$, $1 < p < \infty$, and if u is the solution of (7.10), then we multiply the equation by $\sigma_p(u(x))$, where $\sigma_p(s) = |s|^{p-1} \text{sgn}(s)$ is the monotone duality map for $L^p(G)$, and integrate to obtain $\|u\|_{L^p}^p \leq \|f\|_{L^p} \|u\|_{L^p}^{p-1}$, hence, $\|u\|_{L^p(G)} \leq \|f\|_{L^p(G)}$.

Similarly, multiply by $\sigma_p(v) \in (\sigma_p \circ \alpha)(u)$ and note that $\sigma_p \circ \alpha$ is maximal monotone so from Proposition II.9.3 we obtain $\|v\|_{L^p(G)} \leq \|f\|_{L^p(G)}$. We conclude that if $f \in L^p(G)$, the solution u satisfies

$$u \in D(A), Au \in L^p(G), \text{ and for some } v \in L^p(G) v(x) \in \alpha(u(x)), \quad \text{a.e. } x \in G.$$

We define $D(A + \alpha)_p$ to be the set of all such u . Suppose $f_1, f_2 \in L^p(G)$, $\varepsilon > 0$, and u_j, v_j are the corresponding solutions of (7.10) for $j = 1, 2$. Subtracting the equations, multiplying by $\sigma_p(u_1 - u_2)$, and integrating the products shows that $\|u_1 - u_2\|_{L^p(G)} \leq \|f_1 - f_2\|_{L^p(G)}$. This shows that the operator $A + \alpha$ with domain $D(A + \alpha)_p$ is m -accretive on $L^p(G)$.

Example. Let G be a bounded domain in $\mathbb{R}^n$, $1 < p < \infty$, and let $A = A_1$, the operator (7.9) in $L^1(G)$ constructed in Proposition II.9.1 with domain $D(A) = \{u \in W_0^{1,1}(G) : A_1 u \in L^1(G)\}$. Then $u \in L^p(G) \cap D(A)$ and $A_1 u \in L^p(G)$ imply that $u \in W_0^{1,p}(G) \cap W^{2,p}(G)$. According to Theorem 7.1, for each $u_0 \in D(A + \alpha)_p$ and $f \in W^{1,1}(0, T; L^p(G))$ there is a unique solution $u \in W^{1,\infty}(0, T; L^p(G))$ of the *semilinear parabolic* initial-boundary-value problems

$$(7.11.a) \quad \frac{\partial u(t)}{\partial t} + A_1 u(t) + v(t) = f, \quad v(t) \in \alpha(u(t)) \text{ in } L^p(G)$$

$$(7.11.b) \quad u(t) \in W_0^{1,p}(G) \cap W^{2,p}(G), \quad \text{a.e. } t \in (0, T),$$

$$(7.11.c) \quad u(0) = u_0 \text{ in } L^p(G).$$

This provides a rather strong notion of solution, even though α is very general. One can similarly apply Proposition 7.3 to get periodic solutions of (7.11.a) and (7.11.b) if $a(x)I + \alpha(\cdot)$ is strongly monotone, *i.e.*, there is an $\omega > 0$ for which

$$a(x)(r - s)^2 + (\xi - \eta)(r - s) \geq \omega(r - s)^2, \quad [r, \xi], [s, \eta] \in \alpha.$$

IV.8. The Cauchy Problem in General Banach Space

Let A be an operator in the Banach space X , possibly multi-valued, and let $f \in L^1(a, b; X)$. We shall consider the evolution equation

$$(8.1) \quad u'(t) + A(u(t)) \ni f(t), \quad a < t < b.$$

DEFINITION. An ε -solution of (8.1) is a discretization

$$\mathcal{D} \equiv \{a = t_0 < t_1 < \dots < t_N = b; f_1, f_2, \dots, f_N \in X\}$$

and a step function $v(t) = \begin{cases} v_0, & t = t_0 \\ v_j, & t \in (t_{j-1}, t_j] \end{cases}$ for which

$$t_j - t_{j-1} \leq \varepsilon \text{ for } 1 \leq j \leq N,$$

$$\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|f(t) - f_j\| dt < \varepsilon, \text{ and}$$

$$\frac{v_j - v_{j-1}}{t_j - t_{j-1}} + A(v_j) \ni f_j, \quad 1 \leq j \leq N.$$

We note that if A is m -accretive, then v is determined by $\mathcal{D}$.

DEFINITION. A C^0 -solution of (8.1) is a $u \in C([a, b], X)$ such that for every $\varepsilon > 0$ there is an ε -solution $\mathcal{D}, v$ of (8.1) with

$$\|u(t) - v(t)\| \leq \varepsilon, \quad a \leq t \leq b.$$

PROPOSITION 8.1.

- (a) If u is a C^0 -solution on $[a, b]$ and $[c, d] \subset [a, b]$, then u is a C^0 -solution on $[c, d]$.
- (b) If $u \in C([a, b], X)$ is a C^0 -solution on $[a, c]$ and on $[c, b]$, then u is a C^0 -solution on $[a, b]$.
- (c) If u is a C^0 -solution in X , and X is continuously embedded in the Banach space Y , then u is a C^0 -solution in Y .
- (d) If each u_n is a C^0 -solution of (8.1) with f_n , if $u_n \rightarrow u$ in $C([a, b], X)$ and $f_n \rightarrow f$ in $L^1(a, b; X)$, then u is a C^0 -solution of (8.1) with f .
- (e) If u is a C^0 -solution of (8.1) with A_1 , and $A_1 \subset A_2$, then u is a C^0 -solution of (8.1) with A_2 .
- (f) Let $\bar{A}$ be the closure of A in $X \times X$. If u is a C^0 -solution of (8.1) with $\bar{A}$, then u is a C^0 -solution of (8.1) with A .

PROOF. Parts (a) through (e) are straightforward. To prove (f), let $\bar{v}(t)$ be an ε -solution, i.e.,

$$\frac{\bar{v}_j - \bar{v}_{j-1}}{t_j - t_{j-1}} + \bar{A}(\bar{v}_j) \ni \bar{f}_j, \quad \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|f(t) - \bar{f}_j\| dt < \varepsilon.$$

From the definition of $\bar{A}$ we obtain $[v_j, w_j] \in A$ such that

$$\|v_j - \bar{v}_j\| < \varepsilon, \quad \left\| w_j - \left(\bar{f}_j - \frac{\bar{v}_j - \bar{v}_{j-1}}{t_j - t_{j-1}} \right) \right\| < \varepsilon.$$

Set $f_j \equiv w_j + \frac{v_j - v_{j-1}}{t_j - t_{j-1}}$. Then we have

$$\|f_j - \bar{f}_j\| \leq \|w_j - \bar{f}_j + \frac{\bar{v}_j - \bar{v}_{j-1}}{t_j - t_{j-1}}\| + \left\| \frac{v_j - v_{j-1} - (\bar{v}_j - \bar{v}_{j-1})}{t_j - t_{j-1}} \right\| < \varepsilon + \frac{2\varepsilon}{t_j - t_{j-1}},$$

and this gives

$$\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|f_j - \bar{f}_j\| dt < \varepsilon(b-a) + 2\varepsilon,$$

so we have

$$\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|f(t) - f_j\| dt < \varepsilon(b-a) + 3\varepsilon.$$

If we have $\|u(t) - \bar{v}(t)\| < \varepsilon$, then $\|u(t) - v(t)\| < 2\varepsilon$. Since ε is arbitrary, we are done. $\square$

DEFINITION. A *strong solution* of (8.1) is a $u \in W^{1,1}(a, b; X)$ such that (8.1) holds at a.e. $t \in (a, b)$.

We note that from the definition of $L^1(a, b; X)$ it follows that any $g \in L^1(a, b; X)$ is the L^1 -limit of measurable step functions. Specifically, for any $\varepsilon > 0$ there is a partition $\{t_j\}$ with each $t_j - t_{j-1} < \varepsilon$ and selections $\tau_j \in [t_{j-1}, t_j]$ with

$$\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|g(t) - g_j\| dt < \varepsilon, \quad g_j = g(\tau_j), \quad 1 \leq j \leq N.$$

Moreover, we can achieve this with $\tau_j = t_j$.

PROPOSITION 8.2. *Every strong solution is a C^0 -solution.*

PROOF. Let $\varepsilon > 0$. By the preceding remark with $g = f - u'$, we choose a partition $\{t_j\}$ and selections $\{f_j\}, \{w_j\}$ in X with

$$\sum_{j=1}^N \int_{t_{j-1}}^{t_j} (\|f(t) - f_j\| + \|u'(t) - w_j\|) dt < \varepsilon, \quad w_j + A(u(t_j)) \ni f_j, \quad 1 \leq j \leq N.$$

Set $v_j = u(t_j)$ and

$$g_j = f_j + \frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} (u'(t) - w_j) dt.$$

Then we have

$$\frac{v_j - v_{j-1}}{t_j - t_{j-1}} + A(v_j) \ni g_j, \quad 1 \leq j \leq N,$$

and

$$\int_{t_{j-1}}^{t_j} \|f(t) - g_j\| dt \leq \int_{t_{j-1}}^{t_j} \|f(t) - f_j\| dt + \int_{t_{j-1}}^{t_j} \|u'(t) - w_j\| dt < \varepsilon,$$

so v is an ε -solution on $\mathcal{D} = \{\{t_j\}, \{g_j\}\}$. Also, since u is uniformly continuous,

$$\|u(t) - v(t)\| = \|u(t) - u(t_i)\| \rightarrow 0, \quad t_{j-1} \leq t \leq t_j,$$

uniformly in t, j as the mesh converges to zero, so u is a C^0 -solution. $\square$

We have previously considered the differentiability of the norm, $\varphi(x) = \|x\|$, in a general Banach space. For example, for Lemma II.8.2 we observed that the map $t \mapsto \frac{1}{t} (\varphi(x + ty) - \varphi(x))$ is monotone in $t > 0$ and it is bounded,

$$-\|y\| \leq \frac{1}{t} (\varphi(x + ty) - \varphi(x)) \leq \|y\| ,$$

so the *directional derivative*

$$\varphi'(x, y) \equiv \lim_{t \rightarrow 0^+} \frac{\|x + ty\| - \|x\|}{t} = \inf_{t > 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists at every x and in every direction y . Moreover, in Proposition II.8.4 we noted that $\varphi'(x, y) \geq 0$ if and only if

$$\|x\| \leq \|x + ty\| , \quad t > 0 ,$$

so it follows that the operator A on X is accretive if and only if for $[u_j, v_j] \in A, j = 1, 2$, we have

$$\varphi'(u_1 - u_2, v_1 - v_2) \geq 0 .$$

Additional properties of φ' are collected here.

LEMMA 8.1.

- (a) $\varphi'(x, ax) = a\|x\|, a \in \mathbb{R}$, and
 $\varphi'(ax, y) = \varphi'(x, y), \varphi'(x, ay) = a\varphi'(x, y), a > 0$.
- (b) $\varphi'(x, y_1 + y_2) \leq \varphi'(x, y_1) + \varphi'(x, y_2)$,
 $|\varphi'(x, y_1) - \varphi'(x, y_2)| \leq \|y_1 - y_2\|$.
- (c) $\varphi'(x_1, x_2 - x_1) \leq \|x_2\| - \|x_1\| \leq -\varphi'(x_2, x_1 - x_2)$.
- (d) If $x_n \rightarrow x, y_n \rightarrow y$ in X , then

$$\limsup_{n \rightarrow \infty} \varphi'(x_n, y_n) \leq \varphi'(x, y) .$$

- (e) $-\|y\| \leq -\varphi'(x, -y) \leq \varphi'(x, y) \leq \|y\|$.

PROOF. Parts (a), (b) and (c) follow directly from the definition of φ' . For (d), note that for $t > 0$ we have

$$\varphi'(x_n, y_n) \leq \frac{1}{t} (\|x_n + ty_n\| - \|x_n\|)$$

so we obtain

$$\limsup_{n \rightarrow \infty} \varphi'(x_n, y_n) \leq \frac{1}{t} (\|x + ty\| - \|x\|) , \quad t > 0 .$$

Finally, part (e) follows from the successive estimates

$$\begin{aligned} 2\|x\| &\leq \|x + ty\| + \|x - ty\| , \\ -\|y\| &\leq \frac{\|x\| - \|x - ty\|}{t} \leq \frac{\|x + ty\| - \|x\|}{t} \leq \|y\| . \end{aligned} \quad \square$$

We note that the lower limit

$$\lim_{t \rightarrow 0^-} \frac{1}{t} (\|x + ty\| - \|x\|) = \lim_{t \rightarrow 0^+} \frac{1}{t} (\|x\| - \|x - ty\|) = -\varphi'(x, -y)$$

is the *left derivative* and this corresponds to estimates that are *stronger* than accretive. Finally, corresponding to Proposition II.8.5 we have the following.

LEMMA 8.2. *If $u : [0, T] \rightarrow X$ is right-differentiable (left-differentiable) at $t \in (0, T)$, then $\|u(\cdot)\|$ is also right (respectively, left)-differentiable at t and*

$$D^+ \|u(t)\| = \varphi'(u(t), D^+ u(t)) ,$$

respectively, $D^- \|u(t)\| = -\varphi'(u(t), -D^- u(t))$.

COROLLARY 8.1. *If $u \in W^{1,1}(0, T; X)$, then $\|u(\cdot)\| \in W^{1,1}(0, T)$ and*

$$\frac{d}{dt} \|u(t)\| = \varphi'(u(t), u'(t)) = -\varphi'(u(t), -u'(t)) , \quad \text{a.e. } t \in (0, T) .$$

PROOF. A composition of Lipschitz and absolutely continuous functions is then absolutely continuous, hence a.e. differentiable, and there the left and right derivatives agree. $\square$

This leads to the fundamental a-priori estimate for strong solutions of (8.1).

PROPOSITION 8.3. *Assume A is accretive on X and that for $j = 1, 2$ we have*

$$\begin{aligned} u_j &\in W^{1,1}(0, T; X) , \quad f_j \in L^1(0, T; X) \\ u'_j(t) + A(u_j(t)) &\ni f_j(t) , \quad \text{a.e. } t \in (0, T) . \end{aligned}$$

Then $\|u_1(\cdot) - u_2(\cdot)\| \in W^{1,1}(0, T)$ and

$$(8.2) \quad \frac{d}{dt} \|u_1(t) - u_2(t)\| \leq \varphi'(u_1(t) - u_2(t), f_1(t) - f_2(t)) , \quad \text{a.e. } t \in (0, T) .$$

PROOF. Set $v_j(t) = f_j(t) - u'_j(t) \in A(u_j(t))$ so that $\varphi'(u_1(t) - u_2(t), v_1(t) - v_2(t)) \geq 0$. From Corollary 8.1 and Lemma 8.1(b) we obtain

$$\begin{aligned} \frac{d}{dt} \|u_1(t) - u_2(t)\| &= -\varphi'(u_1(t) - u_2(t), -(u'_1(t) - u'_2(t))) \\ &\leq \varphi'(u_1(t) - u_2(t), f_1(t) - f_2(t)) \\ &\quad - \varphi'(u_1(t) - u_2(t), v_1(t) - v_2(t)) . \end{aligned}$$

a.e. on $(0, T)$, so the estimate follows. $\square$

COROLLARY 8.2. *If u_1, u_2 are strong solutions of (8.1) on $[0, T]$ with data f_1, f_2 , respectively, then*

$$\|u_1(t) - u_2(t)\| \leq \|u_1(0) - u_2(0)\| + \int_0^t \|f_1(s) - f_2(s)\| ds , \quad 0 \leq t \leq T .$$

We can obtain a *discrete* version of (8.2) in the following form.

PROPOSITION 8.4. *Assume A is accretive on X and that an ε -solution of (8.1) is given by*

$$\begin{aligned} v(t) &= v_j , \\ t_{j-1} &< t \leq t_j , \\ \frac{v_j - v_{j-1}}{t_j - t_{j-1}} + A(v_j) &\ni f_j , \quad 1 \leq j \leq N . \end{aligned}$$

Then we have for each $[x, y] \in A$

$$(8.3) \quad \frac{\|v_j - x\| - \|v_{j-1} - x\|}{t_j - t_{j-1}} \leq \varphi'(v_j - x, f_j - y), \quad 1 \leq j \leq N.$$

PROOF. From Lemma 8.1 we have successively

$$\begin{aligned} \frac{\|v_j - x\| - \|v_{j-1} - x\|}{t_j - t_{j-1}} &\leq \frac{-\varphi'(v_j - x, v_{j-1} - v_i)}{t_j - t_{j-1}} \\ &= -\varphi'(v_j - x, \frac{v_{j-1} - v_j}{t_j - t_{j-1}} + f_j - y - (f_j - y)) \\ &\leq -\varphi'\left(v_j - x, \left(\frac{v_{j-1} - v_j}{t_j - t_{j-1}} + f_j\right) - y\right) + \varphi'(v_j - x, f_j - y) \\ &\leq 0 + \varphi'(v_j - x, f_j - y). \quad \square \end{aligned}$$

Next we develop an *integral* version of (8.2).

LEMMA 8.3. For $\varepsilon > 0$ let there be given a pair of real-valued step functions

$$V^\varepsilon(t) = V_j^\varepsilon, F^\varepsilon(t) = F_j^\varepsilon, \quad t_{j-1} < t \leq t_j,$$

where the mesh of the partition $0 < t_1 < t_2 < \dots < t_{N_\varepsilon} = T$ goes to zero with ε . Assume $V_\varepsilon \rightarrow V$ in $C[0, T]$, $|F^\varepsilon| \leq g \in L^1(0, T)$, $F(t) = \limsup_{\varepsilon \rightarrow 0} F^\varepsilon(t)$, and that

$$\frac{V_j^\varepsilon - V_{j-1}^\varepsilon}{t_j - t_{j-1}} \leq F_j^\varepsilon, \quad 1 \leq j \leq N_\varepsilon.$$

Then we have

$$V(t) - V(s) \leq \int_s^t F, \quad 0 \leq s \leq t \leq T.$$

PROOF. By an easy induction there follows

$$V_m^\varepsilon - V_n^\varepsilon \leq \sum_{j=n+1}^m F_j^\varepsilon(t_j - t_{j-1}), \quad 1 \leq n \leq m \leq N_\varepsilon,$$

so the desired estimate follows by dominated convergence. $\square$

Now apply Lemma 8.3 to (8.3). Since φ' is upper-semi-continuous this leads to the following.

COROLLARY 8.3. Assume A is accretive on X , $f \in L^1(0, T; X)$ and u is a C^0 -solution of (8.1). Then for each $[x, y] \in A$ we have

$$(8.4) \quad \|u(t) - x\| \leq \|u(s) - x\| + \int_s^t \varphi'(u(\tau) - x, f(\tau) - y) d\tau, \quad 0 \leq s \leq t \leq T.$$

PROPOSITION 8.5. Assume A is accretive on X , $f_j \in L^1(0, T; X)$, and u_j is a C^0 -solution of (8.1) with f_j for $j = 1, 2$. Then we have

$$\begin{aligned}
 (8.5) \quad & \int_a^b (\|u_1(t) - u_2(\sigma)\| - \|u_1(s) - u_2(\sigma)\|) d\sigma \\
 & + \int_s^t (\|u_1(\tau) - u_2(b)\| - \|u_1(\tau) - u_2(a)\|) d\tau \\
 & \leq \int_a^b \int_s^t \varphi'(u_1(\tau) - u_2(\sigma), f_2(\tau) - f_2(\sigma)) d\tau d\sigma, \\
 & \text{for } 0 \leq s < t < T, 0 \leq a < b \leq T.
 \end{aligned}$$

PROOF. Choose $\mathcal{D} = \{0 = t_0 < t_1 < \dots < t_N = T; f_1, \dots, f_N \in X\}$ and a corresponding ε -solution $v(t) = v_j$ on $(t_{j-1}, t_j]$ for the equation

$$u'_2 + A(u_2) \ni f_2.$$

From Corollary 8.3 with $u = u_1$ and $f = f_1$ we obtain from (8.4)

$$\begin{aligned}
 \|u_1(t) - v_j\| - \|u_1(s) - v_j\| & \leq \int_s^t \varphi' \left(u_1(\tau) - v_j, f_1(\tau) - f_j + \frac{v_j - v_{j-1}}{t_j - t_{j-1}} \right) d\tau \\
 & \leq \int_s^t \left(\varphi' \left(u_1(\tau) - v_j, f_1(\tau) - f_j \right) + \varphi' \left(u_1(\tau) - v_j, \frac{v_j - v_{j-1}}{t_j - t_{j-1}} \right) \right) d\tau.
 \end{aligned}$$

The latter term in this integrand is bounded by

$$\frac{\|u_1(\tau) - v_{j-1}\| - \|u_1(\tau) - v_j\|}{t_j - t_{j-1}},$$

so we obtain

$$\begin{aligned}
 \|u_1(t) - v_j\| - \|u_1(s) - v_j\| & + \int_s^t \frac{\|u_1(\tau) - v_j\| - \|u_1(\tau) - v_{j-1}\|}{t_j - t_{j-1}} d\tau \\
 & \leq \int_s^t \varphi'(u_1(\tau) - v_j, f_1(\tau) - f_j) d\tau, \\
 0 \leq s < t \leq T, 1 \leq j \leq N.
 \end{aligned}$$

Multiply by $t_j - t_{j-1}$ and sum for $j = n+1, n+2, \dots, m$ to obtain

$$\begin{aligned}
 & \int_{t_n}^{t_m} (\|u_1(t) - v(\sigma)\| - \|u_1(s) - v(\sigma)\|) d\sigma \\
 & + \int_s^t (\|u_1(\tau) - v(t_m)\| - \|u_1(\tau) - v(t_n)\|) d\tau \\
 & \leq \int_{t_n}^{t_m} \int_s^t \varphi'(u_1(\tau) - v(\sigma), f_1(\tau) - f(\sigma)) d\tau d\sigma
 \end{aligned}$$

where $f(\sigma) = f_j$ for $\sigma \in (t_{j-1}, t_j]$. Letting $\varepsilon \rightarrow 0$ with $v \equiv v_\varepsilon \rightarrow u_2$ and $f = f_\varepsilon \rightarrow f_2$ leads to (8.5). $\square$

REMARK. Define the functions

$$u(t, s) \equiv \begin{cases} \|u_1(t) - u_2(s)\|, & 0 \leq s \leq t \leq T \\ 0, & 0 \leq t \leq s \leq T \end{cases}$$

$$F(t, s) \equiv \begin{cases} \varphi'(u_1(t) - u_2(s), f_1(t) - f_2(s)), & 0 \leq s \leq t \leq T, \\ 0, & 0 \leq t \leq s \leq T \end{cases}$$

for which we can write (8.5) in the form

$$(8.5') \quad \int_a^b (u(t, \sigma) - u(s, \sigma)) d\sigma + \int_s^t (u(\tau, b) - u(\tau, a)) d\tau \leq \int_a^b \int_s^t F(\tau, \sigma) d\tau d\sigma.$$

If it were true that $u(\cdot, \cdot)$ is absolutely continuous in each variable, then this would imply that

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial s} \leq F(t, s),$$

hence, $\frac{d}{d\sigma} u(\sigma, \sigma) \leq F(\sigma, \sigma)$ and then

$$u(t, t) \leq u(s, s) + \int_s^t F(\sigma, \sigma) d\sigma.$$

This suggests the proof of the following.

THEOREM 8.1 (BENILAN-CRANDALL-EVANS). *Assume that A is accretive on X , $f_j \in L^1(0, T; X)$, and that u_j is a C^0 -solution of (8.1) with f_j for $j = 1, 2$. Then we have*

$$(8.6) \quad \|u_1(t) - u_2(t)\| \leq \|u_1(s) - u_2(s)\| + \int_s^t \varphi'(u_1(\sigma) - u_2(\sigma), f_1(\sigma) - f_2(\sigma)) d\sigma, \quad 0 \leq s \leq t \leq T,$$

and therefore

$$\|u_1 - u_2\|_{C(0, T; X)} \leq \|u_1(0) - u_2(0)\|_X + \|f_1 - f_2\|_{L^1(0, T; X)}.$$

PROOF. The plan is to obtain (8.6) from (8.5') by *regularization*. For this we choose a

$$\rho \in C_0^\infty(-1, 1) : 0 \leq \rho(x), \int_{-1}^1 \rho(x) dx = 1$$

and define $\rho_\varepsilon(x) = \frac{1}{\varepsilon} \rho(\frac{x}{\varepsilon})$ for $\varepsilon > 0$. In $\mathbb{R}^2$ we define *convolutions*

$$u_\varepsilon(t, \sigma) = \iint_{\mathbb{R}^2} \rho_\varepsilon(\xi) \rho_\varepsilon(\eta) u(t - \xi, \sigma - \eta) d\xi d\eta,$$

$$F_\varepsilon(\tau, \sigma) = \iint_{\mathbb{R}^2} \rho_\varepsilon(\xi) \rho_\varepsilon(\eta) F(\tau - \xi, \sigma - \eta) d\xi d\eta.$$

Then we obtain

$$\begin{aligned}
& \int_a^b (u_\varepsilon(t, \sigma) - u_\varepsilon(s, \sigma)) d\sigma + \int_s^t (u_\varepsilon(\tau, b) - u_\varepsilon(\tau, a)) d\tau \\
&= \iint_{\mathbb{R}^2} \rho_\varepsilon(\xi) \rho_\varepsilon(\eta) \left\{ \int_a^b (u(t - \xi, \sigma - \eta) - u(s - \xi, \sigma - \eta)) d\sigma \right. \\
&\quad \left. + \int_s^t (u(\tau - \xi, b - \eta) - u(\tau - \xi, a - \eta)) d\tau \right\} d\xi d\eta \\
&= \iint_{\mathbb{R}^2} \rho_\varepsilon(\xi) \rho_\varepsilon(\eta) \left\{ \int_{a-\eta}^{b-\eta} (u(t - \xi, \sigma) - u(s - \xi, \sigma)) d\sigma \right. \\
&\quad \left. + \int_{s-\xi}^{t-\xi} (u(\tau, b - \eta) - u(\tau, a - \eta)) d\tau \right\} d\xi d\eta \\
&\leq \iint_{\mathbb{R}^2} \rho_\varepsilon(\xi) \rho_\varepsilon(\eta) \left\{ \int_{a-\eta}^{b-\eta} \int_{s-\xi}^{t-\xi} F(\tau, \sigma) d\tau d\sigma \right\} d\xi d\eta \\
&= \iint_{\mathbb{R}^2} \rho_\varepsilon(\xi) \rho_\varepsilon(\eta) \left\{ \int_a^b \int_s^t F(\tau - \xi, \sigma - \eta) d\tau d\sigma \right\} d\xi d\eta \\
&= \int_a^b \int_s^t F_\varepsilon(\tau, \sigma) d\tau d\sigma , \\
&\quad 0 < \varepsilon \leq a \leq b \leq T , \quad 0 < \varepsilon \leq s \leq t \leq T ,
\end{aligned}$$

where the inequality above follows from (8.5'). Now let $\lambda > 0$ and set

$$F^\lambda(t, \sigma) = \frac{1}{\lambda} (\|u_1(t) + \lambda f_1(t) - (u_2(\sigma) + \lambda f_2(\sigma))\| - \|u_1(t) - u_2(\sigma)\|)$$

and note that $F(t, \sigma) \leq F^\lambda(t, \sigma)$, hence, their convolutions satisfy

$$F_\varepsilon(t, \sigma) \leq F_\varepsilon^\lambda(t, \sigma) .$$

By applying our Remark to the estimate above we obtain

$$u_\varepsilon(t, t) \leq u_2(s, s) + \int_s^t F_2^\lambda(\sigma, \sigma) d\sigma , \quad 0 < \varepsilon \leq s \leq t \leq T .$$

Since u is continuous, $u_\varepsilon(t, t) \rightarrow u(t, t)$ as $\varepsilon \rightarrow 0$. Once we show that

$$\int_s^t F_\varepsilon^\lambda(\sigma, \sigma) d\sigma \rightarrow \int_s^t F^\lambda(\sigma, \sigma) d\sigma$$

then (8.6) will follow and we are done.

But F^λ is of the form of the difference of two functions $\|g_1(\tau) - g_2(\sigma)\|$ with $g_j \in L^1(0, T; X)$, so it suffices to consider

$$\begin{aligned}
& \left| \int_s^t (\|g_1(\sigma) - g_2(\sigma)\|_\varepsilon - \|g_1(\sigma) - g_2(\sigma)\|) d\sigma \right| \\
&= \left| \iint_{\mathbb{R}^2} \int_s^t \rho_\varepsilon(\xi) \rho_\varepsilon(\eta) (\|g_1(\sigma - \xi) - g_2(\sigma - \eta)\| - \|g_1(\sigma) - g_2(\sigma)\|) d\sigma d\xi d\eta \right| \\
&\leq \iint_{\mathbb{R}^2} \int_s^t \rho_\varepsilon(\xi) \rho_\varepsilon(\eta) \|g_1(\sigma - \xi) - g_1(\sigma) - g_2(\sigma - \eta) + g_2(\sigma)\| d\sigma d\xi d\eta \\
&\leq \int_{\mathbb{R}} \int_s^t \rho_\varepsilon(\xi) \|g_1(\sigma - \xi) - g_1(\sigma)\| d\sigma d\xi \\
&\quad + \int_{\mathbb{R}} \int_s^t \rho_\varepsilon(\eta) \|g_2(\sigma - \eta) - g_2(\sigma)\| d\sigma d\eta ,
\end{aligned}$$

and each of these converges to zero by standard results on the mollifier in L^1 . $\square$

We turn now to the question of *existence* of a C^0 -solution of (8.1) and begin with the Cauchy problem for the corresponding homogeneous equation

$$(8.7) \quad u'(t) + A(u(t)) \ni 0, \quad 0 < t < T, \quad u(0) = u_0$$

with m -accretive A and $u_0 \in D(A)$. For this case we shall show that the linear interpolants of ε -solutions for a uniform partition of $[0, T]$ are Cauchy in $C(0, T; X)$. The non-homogeneous equation will be easily resolved then by Proposition 8.1 applied to step-functions approximating a given $f \in L^1(0, T; X)$.

Given $T > 0$ and an integer $M \geq 1$, consider the sequence defined by

$$u_m = (I + sA)^{-m} u_0, \quad 0 \leq m \leq M, \quad s = T/M.$$

That is, we have equivalently

$$u_m + sa_m = u_{m-1}, \quad a_m \in A(u_m), \quad 1 \leq m \leq M.$$

Let $a_0 \in A(u_0)$. The following two estimates are basic.

LEMMA 8.4 (LIPSCHITZ).

$$\|u_m - u_{m-1}\| \leq s\|a_0\|, \quad \|u_m - u_0\| \leq ms\|a_0\|, \quad m = 1, 2, \dots, M.$$

PROOF. For $m = 1$ we note

$$u_1 - u_0 + s(a_1 - a_0) = -sa_0$$

so $\|u_1 - u_0\| \leq s\|a_0\|$. For $m > 1$ we have

$$u_m - u_{m-1} + s(a_m - a_{m-1}) = u_{m-1} - u_{m-2}$$

from which $\|u_m - u_{m-1}\| \leq \|u_{m-1} - u_{m-2}\|$ follows by accretiveness again. $\square$

LEMMA 8.5 (COMPARISON). *Let u_m be as above and*

$$v_n = (I + \tau A)^{-n} u_0, \quad 0 \leq n \leq N, \quad \tau = T/N,$$

where $N > M$, hence $\theta \equiv \tau/s \in (0, 1)$. Then

$$\|u_m - v_n\| \leq \theta \|u_{m-1} - v_{n-1}\| + (1 - \theta) \|u_m - v_{n-1}\|, \quad 1 \leq m \leq M, \quad 1 \leq n \leq N.$$

PROOF. From the inclusions

$$u_m - v_n \in u_{m-1} - v_{n-1} - sAu_m + \tau Av_n = u_{m-1} - v_{n-1} - \tau(Au_m - Av_n) - (s - \tau)Au_m$$

and

$$(s - \tau)Au_m \ni \frac{s - \tau}{s}(u_{m-1} - u_m),$$

we have

$$\begin{aligned} u_m - v_n + \tau(Au_m - Av_n) &\ni u_{m-1} - v_{n-1} - (1 - \theta)(u_{m-1} - u_m) \\ &= \theta(u_{m-1} - v_{n-1}) + (1 - \theta)(u_m - v_{n-1}). \end{aligned}$$

This yields the desired estimate. $\square$

Now set $a_{m,n} \equiv \|u_m - v_n\|$ and $K = \|a_0\|$. According to the preceding estimates, we are in the situation of the following.

PROPOSITION 8.6. *Consider the rectangular grid*

$$\mathcal{R} \equiv \{0, 1, 2, \dots, M\} \times \{0, 1, 2, \dots, N\}$$

with boundary grid

$$\partial\mathcal{R} \equiv \{0\} \times \{0, 1, \dots, N\} \cup \{0, 1, \dots, M\} \times \{0\}.$$

If the function $a : \mathcal{R} \rightarrow \mathbb{R}^+$ satisfies

$$a_{m,0} \leq msK, \quad 0 \leq m \leq M,$$

$$a_{0,n} \leq n\tau K, \quad 0 \leq n \leq N, \text{ and}$$

$$a_{m,n} \leq \theta a_{m-1,n-1} + (1 - \theta)a_{m,n-1}, \quad 0 < m \leq M, \quad 0 < n \leq N,$$

with $\theta = \tau/s \in (0, 1)$, then

$$(8.8) \quad a_{m,n} \leq K [n\tau(s - \tau) + (ms - n\tau)^2]^{1/2}.$$

Let $u, v \in C(0, T; X)$ be the piece-wise linear functions for which $u(ms) = u_m$ and $v(n\tau) = v_n$. Then for each $t \in [0, T]$ we have

$$t = ms + \xi = n\tau + \eta, \quad 0 \leq \xi < s, \quad 0 \leq \eta < \tau,$$

and furthermore

$$u(t) = u_m + (u_{m+1} - u_m) \frac{\xi}{s}, \quad v(t) = v_n + (v_{n+1} - v_n) \frac{\eta}{\tau}.$$

It follows, then, from (8.8) that

$$\begin{aligned} \|u(t) - v(t)\| &\leq \|u_m - v_n\| + \|u_{m+1} - u_m\| + \|v_{n+1} - v_n\| \\ (8.9) \quad &\leq \|a_0\| [T(s - \tau) + (\xi - \eta)^2]^{1/2} + \|a_0\| (s + \tau) \\ &\leq \|a_0\| [T(s - \tau) + s^2]^{1/2} + \|a_0\| (s + \tau), \end{aligned}$$

and this goes uniformly to zero as $s = \frac{1}{M} \geq \tau = \frac{1}{N} \rightarrow 0$. Thus, the piece-wise linear interpolants of $\{u_m\}$ converge uniformly to some $u \in C(0, T; X)$. Since the corresponding step functions converge uniformly to u , it is a C^0 -solution of (8.7), and it is Lipschitz continuous with

$$\|u(t) - u(s)\| \leq |t - s| \inf\{\|a\| : a \in A(u_0)\}.$$

Except for the proof of Proposition 8.6, this gives the following.

PROPOSITION 8.7. *Let A be m -accretive on the Banach space X and $u_0 \in D(A)$. There is a unique $u \in C([0, \infty), X)$ such that, for every $T > 0$, u is a C^0 -solution of (8.7) on $[0, T]$ and u is Lipschitz.*

The following is an immediate extension of Proposition 8.7.

THEOREM 8.2 (CRANDALL-LIGGETT). *Assume A is accretive on the Banach space X and that $Rg(I + \lambda A) \supset \overline{D(A)}$ for every $\lambda > 0$. Let $u_0 \in \overline{D(A)}$. Then $u(t) \equiv \lim_{m \rightarrow \infty} (I + \frac{t}{m}A)^{-m}u_0$ exists uniformly on each $[0, T]$, and u is the unique C^0 -solution of (8.7).*

PROOF. For each $\varepsilon > 0$ there is a $[\bar{u}, \bar{v}] \in A$ with $\|\bar{u} - u_0\| < \varepsilon$. Set

$$\bar{u}_m(t) = (I + sA)^{-m}\bar{u}, \quad u_m(t) = (I + sA)^{-m}u_0$$

for $t = ms + \xi$, $0 \leq \xi < s$. Then from (8.9) we obtain

$$\|\bar{u}_m(t) - \bar{u}_n(t)\| \leq \|\bar{v}\| \sqrt{Ts + s^2} = \|\bar{v}\| T \sqrt{\frac{1}{M} + \frac{1}{M^2}}, \quad n \geq m \geq M,$$

so we have

$$\|u_m(t) - u_n(t)\| \leq \|\bar{v}\| T \sqrt{\frac{1}{M} + \frac{1}{M^2}} + 2\varepsilon.$$

This shows $u_m \rightarrow u$ in $C(0, T; X)$. Note that u_m is the $\frac{1}{M}$ -solution for the discretization $\mathcal{D} = \{0, \frac{1}{M}, \frac{2}{M}, \dots, T; 0, \dots, 0\}$. $\square$

REMARK. The function u is not necessarily Lipschitz, but we have the estimate

$$\|u_m(t) - u(t)\| \leq 2\|u_0 - \bar{u}\| + T\|\bar{v}\| \sqrt{\frac{1}{M} + \frac{1}{M^2}}, \quad m \geq M, \quad [\bar{u}, \bar{v}] \in A.$$

COROLLARY 8.4. *Assume A is m -accretive in the Banach space X . For each $u_0 \in \overline{D(A)}$ and $f \in L^1(0, T; X)$ there is a unique C^0 -solution of the Cauchy problem*

$$(8.10) \quad u'(t) + A(u(t)) \ni f(t), \quad 0 < t < T, \quad u(0) = u_0.$$

PROOF. If f is a constant, then $A(\cdot) - f$ is m -accretive and Theorem 8.2 applies. If f is a stepfunction, we get a C^0 -solution on each interval on which f is constant, then Proposition 8.1.b gives a solution on $[0, T]$. Finally, if $f = \lim_{n \rightarrow \infty} f_n$ in $L^1(0, T; X)$ where each f_n is a stepfunction, then the corresponding solutions satisfy $\lim_{n \rightarrow \infty} u_n = u$ in $C([0, T], X)$ by Theorem 8.1, and from Proposition 8.1.d it follows that u is a C^0 -solution. $\square$

We consider next the dependence of the solution u on the operator A . Let's begin with the following notion of convergence of operator.

DEFINITION. Let A and A_n , $n \geq 1$, be operators in X . Then $\{A_n\}$ is *graph-convergent* to A if for every $[u, v] \in A$ there are $[u_n, v_n] \in A_n$, $n \geq 1$, with $[u_n, v_n] \rightarrow [u, v]$ in $X \times X$.

PROPOSITION 8.8. *Let A and each A_n , $n \geq 1$, be m -accretive in X . Then $\{A_n\}$ is graph-convergent to A if and only if $(I + A_n)^{-1}f \rightarrow (I + A)^{-1}f$ for every $f \in X$.*

PROOF. If we have graph-convergence, then let $f \in X$ and set $u_n = (I + A_n)^{-1}f$, $u = (I + A)^{-1}f$, so $[u, f - u] \in A$. Then there exist $[\bar{u}_n, \bar{v}_n] \in A_n$ with $\bar{u}_n \rightarrow u$, $\bar{v}_n \rightarrow f - u$, and $\bar{u}_n = (I + A_n)^{-1}(\bar{u}_n + \bar{v}_n)$. By accretivity of A ,

$$\|\bar{u}_n - u_n\| \leq \|\bar{u}_n + \bar{v}_n - f\| \rightarrow 0.$$

Since $\bar{u}_n \rightarrow u$ we have $u_n \rightarrow u$.

Conversely, let $[u, v] \in A$, hence, $u = (I + A)^{-1}(u + v)$, and set $u_n \equiv (I + A_n)^{-1}(u + v)$. Then $u_n \rightarrow u$, $v_n \equiv u + v - u_n \rightarrow v$, and $[u_n, v_n] \in A_n$, $n \geq 1$. $\square$

COROLLARY 8.5. *We have*

$$\lim_{n \rightarrow \infty} (I + \lambda A_n)^{-1}f = (I + \lambda A)^{-1}f, \quad f \in X$$

for some $\lambda > 0$ if and only if for all $\lambda > 0$.

COROLLARY 8.6. *For each integer $m \geq 1$,*

$$\lim_{n \rightarrow \infty} (I + A_n)^{-m}f = (I + A)^{-m}f, \quad f \in X.$$

PROOF. Suppose this holds for $m = k \geq 1$. Then

$$\begin{aligned} & \|(I + A_n)^{-k-1}f - (I + A)^{-k-1}f\| = \\ & \|(I + A_n)^{-1}(I + A_n)^{-k}f - (I + A)^{-1}(I + A)^{-k}f\| \\ & \leq \|(I + A_n)^{-k}f - (I + A)^{-k}f\| + \|(I + A_n)^{-1}f - (I + A)^{-1}f\|, \end{aligned}$$

so the result follows by induction on m . $\square$

This leads to the *continuous dependence* of the C^0 -solution of (8.10) on the operator A as well as u_0 and f .

THEOREM 8.3 (BENILAN). *Let A and each A_n , $n \geq 1$, be m -accretive in the Banach space X . Assume $\{A_n\}$ is graph-convergent to A . Assume $f_n \rightarrow f$ in $L^1(0, T; X)$ and let $u_0, u_0^n \in \overline{D(A)}$ with $u_0^n \rightarrow u_0$ in X . Then $u_n \rightarrow u$ in $C([0, T], X)$, where u is the C^0 -solution of (8.10) and each u_n is the corresponding C^0 -solution with A_n, f_n, u_0^n , $n \geq 1$.*

PROOF. Consider first the case $f_n = f = 0$. For $M > 1$ and $s = T/M$, $ms \leq t < (m+1)s$, $m = 0, 1, \dots, M-1$, we obtain

$$\begin{aligned} \|u_n(t) - u(t)\| &\leq \|u_n(t) - (I + sA_n)^{-m}u_0^n\| \\ &\quad + \|(I + sA_n)^{-m}u_0^n - (I + sA_n)^{-m}u_0\| \\ &\quad + \|(I + sA_n)^{-m}u_0 - (I + sA)^{-m}u_0\| \\ &\quad + \|(I + sA)^{-m}u_0 - u(t)\| \\ &\leq \left(2\|u_0^n - u^n\| + \sqrt{\frac{2}{M}}T\|v_n\|\right) + \|u_0^n - u_0\| \\ &\quad + \|(I + sA_n)^{-m}u_0 - (I + sA)^{-m}u_0\| \\ &\quad + \left(2\|u_0 - u\| + \sqrt{\frac{2}{M}}T\|v\|\right) \end{aligned}$$

for any sequence $[u_n, v_n] \in A_n$ and any $[u, v] \in A$. Pick $[u, v]$ as above so that $\|u_0 - u\|$ is small, then pick $[u_n, v_n]$ so that $u_n \rightarrow u$, $v_n \rightarrow v$, hence, for large n , $\|u_0^n - u^n\|$ is small while $\|v_n\|$ is bounded. Choose M so large that all terms are small, except possibly the third. Then letting n get large makes this last one also arbitrarily small by Corollary 8.6. $\square$

We have established the desired result for the case $f_n = f = 0$, and it easily follows for any constant, hence, for any step function $g = f_n = f$. Let v, v_n be the solutions of the corresponding problems for u, u_n but with f, f_n replaced by the step function g . By our preceding remark, $v_n \rightarrow v$ in $C([0, T], X)$. But we have

$$\begin{aligned} \|u_n - u\|_{L^\infty} &\leq \|u_n - v_n\|_{L^\infty} + \|v_n - v\|_{L^\infty} + \|v - u\|_{L^\infty} \\ &\leq \|f_n - g\|_{L^1} + \|v_n - v\|_{L^\infty} + \|g - f\|_{L^1} \end{aligned}$$

and this shows $\limsup_{n \rightarrow \infty} \|u_n - u\|_{L^\infty} \leq 2\|f - g\|_{L^1}$. But g is arbitrary so we have $u_n \rightarrow u$ in $C([0, T], X)$.

Finally, we return to the proof of Proposition 8.6. Let $\mathcal{R}$ denote the rectangular grid with boundary $\partial\mathcal{R}$ as before.

LEMMA 8.6. *If the functions a, b from $\mathcal{R}$ to $\mathbb{R}^+$ satisfy*

$$(8.11.a) \quad a_{m,n} \leq \theta a_{m-1,n-1} + (1-\theta)a_{m,n-1},$$

$$(8.11.b) \quad b_{m,n} = \theta b_{m-1,n-1} + (1-\theta)b_{m,n-1}, \quad 1 \leq m \leq M, \quad 1 \leq n \leq N,$$

and $a \leq b$ on $\partial\mathcal{R}$, then $a \leq b$ on $\mathcal{R}$.

PROOF. This is proved by induction on $\max(M, N)$, the size of the grid $\mathcal{R}$. If $\max(M, N) = 1$ then the only value in question is $a_{1,1}$, all other points being on $\partial\mathcal{R}$. However, this follows immediately from the calculation,

$$\begin{aligned} a_{1,1} &\leq \theta a_{0,0} + (1-\theta)a_{0,1} \\ &\leq \theta b_{0,0} + (1-\theta)b_{1,0} = b_{1,1}. \end{aligned}$$

Assuming the result for grids of size less than $\max(M, N)$ it suffices to show the result for the column $\{1\} \times \{1, 2, \dots, N\}$ and row $\{1, 2, \dots, M\} \times \{1\}$, since this is

the boundary of the smaller $(M - 1) \times (N - 1)$ grid $\{1, 2, \dots, M\} \times \{1, 2, \dots, N\}$. The calculation

$$\begin{aligned} a_{m+1,1} &\leq \theta a_{m,0} + (1 - \theta) a_{m+1,0} \\ &\leq \theta b_{m,0} + (1 - \theta) b_{m+1,0} = b_{m+1,1} , \end{aligned}$$

shows that the first row satisfies the result. The first column is shown to satisfy the inequality by observing that $a_{1,1} \leq b_{1,1}$, since it is also in the first row, and assuming that $a_{1,n} \leq b_{1,n}$ the relation

$$\begin{aligned} a_{1,n+1} &\leq \theta a_{1,n} + (1 - \theta) a_{1,n} \\ &\leq \theta b_{1,n} + (1 - \theta) b_{1,n} = b_{1,n+1} , \end{aligned}$$

shows the next element in the first row satisfies the same inequality. $\square$

The preceding induction argument can also be used to show that given boundary values on $\partial\mathcal{R}$ extend to a function on $\mathcal{R}$ satisfying scheme (8.11.b). Then Lemma 8.6 shows this extension to be unique.

COROLLARY 8.7. *If the functions $a, b : \mathcal{R} \rightarrow \mathbb{R}^+$ satisfy (8.11) and $a^p \leq b$ on $\partial\mathcal{R}$ for some $p \geq 1$, then $a^p \leq b$ on $\mathcal{R}$.*

PROOF. By the convexity of x^p ,

$$\begin{aligned} a_{m+1,n+1}^p &\leq (\theta a_{m,n} + (1 - \theta) a_{m+1,n})^p \\ &\leq \theta a_{m,n}^p + (1 - \theta) a_{m+1,n}^p , \end{aligned}$$

so a^p satisfies (8.11.a). $\square$

LEMMA 8.7. *Let $c_{m,n} = K^2 |ms - n\tau|^2$ on $\partial\mathcal{R}$. Then its extension to all of $\mathcal{R}$ satisfying scheme (8.11.b) satisfies*

$$c_{m,n} \leq K^2 [(n\tau)(s - \tau) + |ms - n\tau|^2] .$$

PROOF. Define $d_{m,n} = c_{m,n} - K^2 |ms - n\tau|^2$ so that

$$\begin{aligned} d_{m+1,n+1} &= a_{m+1,n+1} - K^2 |ms - n\tau|^2 \\ &= \theta a_{m,n} + (1 - \theta) a_{m+1,n} - K^2 |(m+1)s - (n+1)\tau|^2 \\ &= \theta d_{m,n} + (1 - \theta) d_{m+1,n} - K^2 |(m+1)s - (n+1)\tau|^2 \\ &\quad + \theta K^2 |ms - n\tau|^2 + (1 - \theta) K^2 |(m+1)s - n\tau|^2 . \end{aligned}$$

Putting $\beta = (ms - n\tau)$ enables the last three terms in the above to be written

$$\theta |\beta|^2 + (1 - \theta) |\beta + s|^2 - |\beta + (s - \tau)|^2 = \tau(s - \tau) ,$$

where the definition $\theta = \tau/s$ was used to accomplish this simplification. Using this in the first estimate yields

$$d_{m+1,n+1} = \theta d_{m,n} + (1 - \theta) d_{m+1,n} + K^2 \tau(s - \tau)$$

implying

$$\max_{0 \leq m \leq M} d_{m,n+1} \leq \max_{0 \leq m \leq M} d_{m,n} + K^2 \tau(s - \tau) .$$

Summing on n from zero to $n - 1$ gives

$$\begin{aligned} d_{m,n} &\leq \max_{0 \leq m \leq M} d_{m,n} \\ &\leq \max_{0 \leq m \leq M} d_{m,0} + K^2(n\tau)(s - \tau) \\ &\leq K^2(n\tau)(s - \tau) \end{aligned}$$

where the identity $d_{m,0} = 0$ was used in the last step. Inserting the definition of $d_{m,n}$ into the last estimate finishes the proof. $\square$

PROOF OF PROPOSITION 8.6. First check to see that $a_{m,n} \leq K|ms - n\tau|$ for $(m, n) \in \partial\mathcal{R}$. Define $b : \mathcal{R} \rightarrow \mathbb{R}_0^+$ to be the extension satisfying scheme (8.11.b) of the boundary data $b_{m,n} = K|ms - n\tau|$, $(m, n) \in \partial\mathcal{R}$. Then $b^2 = c$ on $\partial\mathcal{R}$ where $c : \mathcal{R} \rightarrow \mathbb{R}_0^+$ is the function defined in Lemma 8.7. Lemmas 8.6 and 8.7 then guarantee

$$\begin{aligned} a_{m,n} &\leq b_{m,n} \\ &\leq (c_{m,n})^{1/2} \\ &\leq K [n\tau(s - \tau) + (ms - n\tau)^2]^{1/2} . \end{aligned} \quad \square$$

IV.9. Evolution Equations in L^1

We present here a collection of examples of the many applications of Theorem 8.2. These include a scalar conservation law, the porous medium equation, and some systems which contain either of them. Each of these equations or systems will be regarded as an abstract Cauchy problem in $L^1(G)$ of the form

$$(9.1) \quad \begin{aligned} u'(t) + \mathbb{A}(u(t)) &\ni f(t) \text{ in } L^1(G), \quad t \in [0, T], \\ u(0) &= u_0, \end{aligned}$$

where $\mathbb{A}$ is the realization of an appropriate m -accretive operator in $L^1(G)$.

The following result is very useful in the construction of these examples, and we repeat it here with a change of notation for reference.

THEOREM II.9.2 (BREZIS-STRAUSS). *Let α be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ and $0 \in \alpha(0)$. Let $A : D(A) \rightarrow L^1(G)$ be linear and satisfy the following:*

- (i) $D(A)$ is dense and $(I + \lambda A)^{-1}$ is a contraction in L^1 for each $\lambda > 0$;
- (ii) $\sup_G (I + \lambda A)^{-1} f \leq (\sup_G f)^+ = \|f^+\|_{L^\infty}$ for $f \in L^1$ and $\lambda > 0$;
- (iii) there is a $c > 0$ such that

$$c\|u\|_{L^1} \leq \|Au\|_{L^1} \quad \text{for } u \in D(A).$$

Then for each $f \in L^1$ there is a unique pair $u \in L^1$, $v \in D(A)$ such that

$$u + Av = f \quad \text{and} \quad v(x) \in \alpha(u(x)) , \quad \text{a.e. } x \in G .$$

If u_1, v_1 and u_2, v_2 are solutions corresponding to f_1, f_2 as above, then

$$\|(u_1 - u_2)^+\|_{L^1} \leq \|(f_1 - f_2)^+\|_{L^1} , \quad \|(u_1 - u_2)^-\|_{L^1} \leq \|(f_1 - f_2)^-\|_{L^1} ,$$

and, hence,

$$\|u_1 - u_2\|_{L^1} \leq \|f_1 - f_2\|_{L^1} .$$

If $f_1 \geq f_2$ a.e. then $u_1 \geq u_2$ a.e. on G .

EXAMPLE 9.A SCALAR CONSERVATION LAW. The first problem to be discussed here is an initial-boundary-value problem for a *scalar conservation law*: find a pair of functions, $u(\cdot, \cdot)$, $v(\cdot, \cdot)$, on $(a, b) \times (0, T)$ which satisfy

$$(9.2) \quad \begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} &= f, \quad v \in \alpha(u) \quad \text{in } (a, b) \times (0, T), \\ v(a, t) &= cv(b, t) \quad \text{for } t \in (0, T), \\ u &= u_0 \quad \text{on } [a, b] \times \{0\}, \end{aligned}$$

where $a < b$ and $0 \leq c < 1$ are given.

The first order spatial operator is the L^1 realization of Example I.4.B. Set $D(A) = \{v \in W^{1,1}(a, b) : v(a) = cv(b)\}$. Define $A = \partial$ on $D(A)$. We show that it satisfies the hypotheses in Theorem II.9.2. It is easy to check that there is a solution $u \in D(A)$ of $(I + \lambda A)(u) = f$ for each $f \in L^1(a, b)$ and $\lambda > 0$. This resolvent equation is a simple problem for an ordinary differential equation, and we can solve it directly. For any such solution, we multiply the equation by $\text{sgn}(u)$ and integrate to get $\|u\|_{L^1(a, b)} \leq \|f\|_{L^1(a, b)}$, so (i) holds. Multiply the identity

$$u(x) - k + \lambda Au(x) = f(x) - k$$

by $\text{sgn}^+(u(x) - k)$, integrate, and set $k = \|f^+\|_{L^\infty(a, b)}$ to obtain

$$\|(u - k)^+\|_{L^1(a, b)} + (u(b) - k)^+ - (cu(b) - k)^+ \leq 0.$$

Since $k \geq 0$, $c \geq 0$ and the *positive part* function $(\cdot)^+$ is monotone, we have

$$\begin{aligned} (u(b) - k)^+ - (cu(b) - k)^+ &\geq (u(b) - k)^+ - (cu(b) - ck)^+ \\ &\geq (1 - c)(u(b) - k)^+ \geq 0, \end{aligned}$$

so $(u - k)^+ = 0$ and we have (ii). To check (iii), we let $Au = f$. Multiply by $\text{sgn}(u)$ and integrate to get

$$|u(x)| \leq |u(a)| + \|f\|_{L^1(a, b)}, \quad a \leq x \leq b.$$

Integrating $Au = f$ and using the boundary condition give $u(a) = \frac{c}{1-c} \|f\|_{L^1(a, b)}$, so with the above we obtain (iii) from

$$|u(x)| \leq \frac{1}{1-c} \|f\|_{L^1(a, b)}, \quad a \leq x \leq b.$$

Theorem II.9.2 asserts that for any maximal monotone α in $\mathbb{R} \times \mathbb{R}$ with $0 \in \alpha(0)$, the operator $\mathbb{A} = A \circ \alpha$ is m -accretive in $L^1(a, b)$. Specifically, if $\lambda > 0$ and $f \in L^1(a, b)$ there is a unique pair

$$u \in L^1(a, b) \quad , \quad v \in W^{1,1}(a, b),$$

for which $v(a) = cv(b)$ and

$$u(x) + \lambda \partial v(x) = f(x) \quad , \quad v(x) \in \alpha(u(x)) \quad , \quad \text{a.e. } x \in (a, b),$$

and the mapping $f \mapsto u$ is an $L^1(a, b)$ -contraction.

From Theorem 8.2 it follows that there is a unique C^0 solution of the initial-boundary-value problem (9.2), and that it is obtained as the limit of solutions of the corresponding problem with the backward difference approximations of the time derivative. It is easy to show by examples that such partial differential equations do

not admit smooth solutions, not even *continuous* solutions, even if $\alpha(\cdot)$ is a smooth invertible function and the initial data $u_0(\cdot)$ is smooth. For the case of a smooth function $\alpha(\cdot)$ the equation has the form

$$\frac{\partial u}{\partial t} + \alpha'(u) \frac{\partial u}{\partial x} = f,$$

where $\alpha'(u)$ is the *signal speed*. Thus, any solution u is constant along a trajectory $x(t)$ with the signal speed $x'(t) = \alpha'(u(x(t), t))$, and this speed depends on the value of the solution, u . In particular, if $\alpha'(u)$ is increasing and positive, then waves move rightward and get steeper until they cease to be functions. However, it is known that the solution obtained here is the *entropy* solution of the scalar conservation law.

EXAMPLE 9.B POROUS MEDIUM EQUATION. We consider the Dirichlet initial-boundary-value problem for the *porous medium equation*: find a pair of functions, $u(\cdot, \cdot)$, $v(\cdot, \cdot)$, on $G \times (0, T)$ for which

$$(9.3.a) \quad u_t - \Delta v = f, \quad v \in \alpha(u) \quad \text{in } G \times (0, T),$$

$$(9.3.b) \quad v = 0 \quad \text{on } \partial G \times (0, T),$$

$$(9.3.c) \quad u = u_0 \quad \text{on } G \times \{0\},$$

where $\alpha(\cdot)$ is a maximal monotone graph as above, G is a bounded domain in $\mathbb{R}^m$, and $T > 0$ denotes the length of the time interval.

We record here a special case of Proposition II.9.1. Define $D(A) = \{v \in W_0^{1,1}(G) : Av \in L^1(G)\}$, where $Av = f \in L^1(G)$ means

$$v \in W_0^{1,1}(G) : \int_G \nabla v \cdot \nabla w = \int_G f w, \quad w \in W_0^{1,\infty}(G).$$

Thus, $Av = -\Delta v \in L^1(G)$ with Dirichlet boundary conditions. Let α be given as above. Then for each $\lambda > 0$ and $f \in L^1(G)$ there is a unique pair

$$u \in L^1(G) \quad , \quad v \in W_0^{1,1}(G)$$

for which the Laplacian $\Delta v \in L^1(G)$, and

$$u(x) - \lambda \Delta v(x) = f(x) \quad , \quad v(x) \in \alpha(u(x)) \quad , \quad \text{a.e. } x \in G.$$

The mapping $f \mapsto u$ is a contraction in $L^1(G)$. The operator $\mathbb{A} = A \circ \alpha$ corresponds to the stationary problem for the *porous medium equation*, and the above shows that it is m -accretive in $L^1(G)$. Thus, Theorem 8.2 gives the existence and uniqueness of the C^0 solution of (9.3) in $L^1(G)$ for each appropriate choice of initial data, u_0 , and source term, f . One can construct similar operators by varying either the linear elliptic part, $-\Delta$, as in Proposition II.9.1, or the boundary conditions. It is not difficult to add perturbations, such as $\beta(u)$, especially if β is continuous and monotone, but more generality leads to some difficulties, especially if both α and β are multi-valued.

There is a wealth of information on the regularity of solutions of the porous medium equation, and it is intimately dependent on the form of the function (or graph), α . One can show by examples that the derivative $\frac{\partial u}{\partial t}$ need not even exist without further assumptions on α . In particular, it is known that if α is Lipschitz continuous, then the component v of the solution of (9.3) is continuous; if α^{-1} is

Lipschitz continuous, then u is continuous and $\frac{\partial u}{\partial t} \in L^2(G)$. The first case occurs in the example of the *Stefan Problem* in which the function has the *degenerate* form $\alpha(s) = s^- + (s - 1)^+$, and the *singular* case arises in problems of *partial saturation* in which $\alpha^{-1}(s) = 1 + (s - 1)^-$. Even the sense in which the partial differential equation is satisfied can be an issue. If α is continuous and u is bounded, then it follows that the equation holds in the sense of generalized functions, i.e., in $\mathcal{D}(G)^*$. If α is onto, then it can be shown that the L^1 solution is also the (strong) H^{-1} -solution and thereby satisfies the partial differential equation in $\mathcal{D}(G)^*$.

We can give an explicit example for the case in which the graph α satisfies $\alpha(0) \supset [0, 1]$ and $\alpha(x) = \{1\}$ for $x \in [0, 1]$. Thus, neither α nor α^{-1} is continuous. It is easy to check that the pair of functions given on $t < \frac{1}{8}$ by

$$\begin{aligned} u(x, t) &= H(x - \sqrt{2t}) - H(x - 1 + \sqrt{2t}) \\ v(x, t) &= \min \frac{x}{\sqrt{2t}}, \frac{1-x}{\sqrt{2t}} \end{aligned}$$

for $x \in G \equiv (0, 1)$, and $u = v = 0$ on $t > \frac{1}{8}$, is the unique solution of (9.3) with $u_0 = 1$ and $f = 0$. Note that this solution u is continuous into $L^1(0, 1)$ and differentiable into $H^{-1}(0, 1)$.

EXAMPLE 9.C EVOLUTION EQUATIONS WITH HYSTERESIS. We consider a system consisting of a parabolic partial differential equation and an ordinary differential equation which are nonlinearly coupled by the difference of the unknowns. This system has the form

$$(9.4.a) \quad \frac{\partial}{\partial t} \alpha(u(x, t)) - \Delta u(x, t) - \gamma(v(x, t) - u(x, t)) \ni f(x, t),$$

$$(9.4.b) \quad \frac{\partial}{\partial t} \beta(v(x, t)) + \gamma(v(x, t) - u(x, t)) \ni g(x, t), \quad x \in G, \quad t \in (0, T],$$

$$(9.4.c) \quad u(s, t) = 0, \quad s \in \partial G,$$

in which $u = u(x, t)$ and $v = v(x, t)$ are functions defined on a bounded domain G in Euclidean space $\mathbb{R}^m$, and $T > 0$. The component (9.4.a) contains a generalized *porous medium equation*, and we make no assumptions of strict monotonicity of $\alpha(\cdot)$. In particular, we allow the degenerate case $\alpha(\cdot) \equiv 0$, and this reduces (9.4) to a *pseudoparabolic* equation.

When each of $\alpha(\cdot)$, $\beta(\cdot)$ and $\gamma(\cdot)$ is a monotone (non-decreasing) function, the inclusion symbols, $\ni$, are replaced by the corresponding equality symbol. Such systems arise in many contexts, for example, in the diffusion of chemicals through a saturated porous medium in which (9.4.b) models adsorption onto immobile non-diffusive sites. In that case, u is the concentration of a chemical species in the fluid which occupies the pores and v is the concentration on the surface of the medium. These are called *first order kinetic models*, and they can be regarded as a degenerate case of corresponding *parallel flow models* which contain an additional term $-\Delta v(x, t)$ in (9.4.b). This diffusion term has been deleted because of the immobility of the concentration in the adsorption sites.

We set $\mathbb{B} = L^1(G) \times L^1(G)$ and define an operator A on $\mathbb{B}$ by $A([u, v]) \ni [f, g]$ if $u \in W_0^{1,1}(G)$, $v \in L^1(G)$ satisfy

$$-\Delta u = g + f, \quad g \in \gamma(v - u).$$

The Dirichlet Laplace, $-\Delta$, was defined in the previous Example 9.B. The operator equation $[u, v] + A([u, v]) \ni [f, g]$ in $\mathbb{B}$ is equivalent to the system

$$\begin{aligned} u &\in W_0^{1,1}(G), \quad u - \Delta u - \xi = f, \\ v &\in L^1(G), \quad v + \xi = g, \quad \xi \in \gamma(v - u), \end{aligned}$$

for some $\xi \in L^1(G)$. This system is equivalent to

$$\begin{aligned} u - \Delta u + (v - u) - (g - u) &= f, \\ v - u + \xi &= g - u, \quad \xi \in \gamma(v - u), \end{aligned}$$

and likewise to

$$\begin{aligned} u - \Delta u + [(I + \gamma)^{-1} - I](g - u) &= f, \\ v - u &= (I + \gamma)^{-1}(g - u). \end{aligned}$$

The second line defines v after u is obtained as the solution of the first line, a monotone Lipschitz perturbation of an m -accretive operator, so this system always has a solution. It is easy to check directly that A is accretive (see below), so it follows that A is m -accretive on $\mathbb{B}$.

Assume for simplicity that the monotone functions $\alpha(\cdot)$ and $\beta(\cdot)$ are Lipschitz continuous. Then the corresponding operators on $L^1(G)$ are continuous and accretive, so it follows that the sum is m -accretive. Thus, for each $\varepsilon > 0$ and each $[f, g] \in \mathbb{B}$ there is a unique solution of the system

$$(9.5.a) \quad \varepsilon u_\varepsilon + \alpha(u_\varepsilon) - \Delta u_\varepsilon - \xi_\varepsilon = f,$$

$$(9.5.b) \quad \varepsilon v_\varepsilon + \beta(v_\varepsilon) + \xi_\varepsilon = g, \quad \xi_\varepsilon \in \gamma(v_\varepsilon - u_\varepsilon).$$

Now define the composite operator $\mathbb{A}$ in $\mathbb{B}$ by $\mathbb{A}([U, V]) \ni [f, g]$ if there exist

$$\begin{aligned} u &\in W_0^{1,1}(G), \quad v \in L^1(G) : \quad U = \alpha(u), \quad V = \beta(v) \\ -\Delta u &= g + f, \quad g \in \gamma(v - u), \end{aligned}$$

that is, $A([u, v]) \ni [f, g]$ in $\mathbb{B}$ and $U = \alpha(u)$, $V = \beta(v)$ in $L^1(G)$. The equation

$$[U, V] + \mathbb{A}([U, V]) \ni [f, g]$$

is just the system

$$\begin{aligned} U &= \alpha(u), \quad \alpha(u) - \Delta u - \xi = f, \\ V &= \beta(v), \quad \beta(v) + \xi = g, \quad \xi \in \gamma(v - u). \end{aligned}$$

In order to show that $\mathbb{A}$ is accretive on $\mathbb{B}$, let $[f_1, g_1], [f_2, g_2] \in \mathbb{B}$ and let $[U_1, V_1], [U_2, V_2]$ be corresponding solutions. Subtract the respective equations to get

$$\begin{aligned} (U_1 - U_2) - \Delta(u_1 - u_2) - (\xi_1 - \xi_2) &= f_1 - f_2, \\ (V_1 - V_2) + (\xi_1 - \xi_2) &= g_1 - g_2, \\ \xi_1 &\in \gamma(v_1 - u_1), \quad \xi_2 \in \gamma(v_2 - u_2). \end{aligned}$$

Multiply the first equation by $\text{sgn}_0(U_1 - U_2 + u_1 - u_2)$, the second by $\text{sgn}_0(V_1 - V_2 + v_1 - v_2)$, integrate over G and add. Note that $\text{sgn}_0(U_1 - U_2 + u_1 - u_2) \in \text{sgn}(U_1 - U_2)$ and $\text{sgn}_0(U_1 - U_2 + u_1 - u_2) \in \text{sgn}(u_1 - u_2)$, and similar inclusions hold for the

second component. Using Theorem 8.2 and the monotonicity of $\alpha(\cdot)$, $\beta(\cdot)$ and $\gamma(\cdot)$, we obtain

$$\|U_1 - U_2\|_{L^1} + \|V_1 - V_2\|_{L^1} \leq \|f_1 - f_2\|_{L^1} + \|g_1 - g_2\|_{L^1}.$$

The same procedure holds with $\mathbb{A}$ replaced by $\varepsilon\mathbb{A}$ for any $\varepsilon > 0$, so $\mathbb{A}$ is accretive in $\mathbb{B}$.

The sum $I + \mathbb{A}$ will be *onto* $\mathbb{B}$ if we can show that for each pair $[f, g] \in \mathbb{B}$ there is a unique solution of the system

$$(9.6) \quad \begin{aligned} \alpha(u) - \Delta u - \xi &= f, \\ \beta(v) + \xi &= g, \quad \xi \in \gamma(v - u). \end{aligned}$$

To this end, for each $\varepsilon > 0$ let u_ε , v_ε be the solution of (9.5). As above, we obtain the estimates

$$\varepsilon\|u_\varepsilon\|_{L^1} + \varepsilon\|v_\varepsilon\|_{L^1} + \|U_\varepsilon\|_{L^1} + \|V_\varepsilon\|_{L^1} \leq \|f\|_{L^1} + \|g\|_{L^1}.$$

From (9.5.b) we get a bound on $\|\xi_\varepsilon\|_{L^1}$ and then from (9.5.a) a bound on $\|\Delta u_\varepsilon\|_{L^1}$ and on $\|u_\varepsilon\|_{L^1}$. We shall assume additionally that β^{-1} is a continuous function and that

$$|s| \leq C(|\beta(s)| + 1), \quad s \in \mathbb{R},$$

for some constant $0 < C$. Then Nemytskii's Theorem II.3.2 shows that β^{-1} gives a continuous operator on L^1 , and this gives a bound on $\|v_\varepsilon\|_{L^1}$. Thus, u_ε , v_ε is a solution of (9.6) with right side given by

$$f_\varepsilon \equiv f - \varepsilon u_\varepsilon \rightarrow f, \quad g_\varepsilon \equiv g - \varepsilon v_\varepsilon \rightarrow g.$$

Since $\mathbb{A}$ is accretive, we obtain, as before, Cauchy-type estimates which imply that

$$U_\varepsilon \rightarrow U, \quad V_\varepsilon \rightarrow V, \quad u_\varepsilon \rightarrow u, \quad \Delta u_\varepsilon \rightarrow \Delta u, \quad \xi_\varepsilon \rightarrow \xi$$

in L^1 , by the continuity of β^{-1} that $v_\varepsilon \rightarrow v$, and then that $U = \alpha(u)$, $V = \beta(v)$, $\xi \in \gamma(v - u)$. Thus, we obtain a solution of (9.6), and it follows that $\mathbb{A}$ is *m*-accretive. From Theorem 8.2 it follows that there is a unique C^0 solution of the initial-boundary-value problem (9.4) for each appropriate choice of initial data, $\alpha(u_0)$, $\beta(v_0)$, and source terms, f , g .

Here we have permitted $\gamma(\cdot)$ to be multi-valued. This generalization includes a very elegant treatment of parabolic problems with variational inequalities. For example, if we define

$$\gamma(s) = \begin{cases} 0, & s < 0, \\ [0, \infty), & s = 0, \end{cases}$$

then the solution of (9.4.b) with $g = 0$ satisfies the *Signorini conditions*

$$u(t) \geq v(t), \quad \frac{\partial}{\partial t} \beta(v(t)) \leq 0, \quad (u(t) - v(t)) \frac{\partial}{\partial t} \beta(v(t)) = 0,$$

in which the exchange term is a *unilateral constraint*. In particular we obtain such equations with *hysteresis* nonlinearities. These appear the form

$$\frac{\partial}{\partial t} (\alpha(u) + \mathcal{H}(u)) - \Delta u = f$$

in which $\mathcal{H}$ denotes a rate-independent *hysteresis* functional, that is, its value depends not only on the current value of the input, u , but also on the *history* of the input, and it does not depend on the rate of the input. For a more interesting

example, a *simple play* hysteresis functional, we choose $\gamma \equiv \text{sgn}^{-1}$. Then for a given input, $u(t)$, the output is the solution $w(t) \equiv \mathcal{H}(u)(t)$ of the equation

$$\frac{\partial}{\partial t} \beta(v(t)) + \gamma(v(t) - u(t)) \ni 0,$$

and it is given by $w(t) = \beta(v(t))$, where $|v(t) - u(t)| \leq 1$ and

$$\begin{cases} w'(t) \geq 0 & \text{if } v(t) = u(t) - 1, \\ w'(t) = 0 & \text{if } |v(t) - u(t)| < 1, \\ w'(t) \leq 0 & \text{if } v(t) = u(t) + 1. \end{cases}$$

By replacing the Dirichlet operator $-\Delta$ in (9.4) by an appropriate first-order operator such as in Example 9.A, we can as well include scalar conservation laws with unilateral constraints or hysteresis functionals.

EXAMPLE 9.D DYNAMIC BOUNDARY CONDITIONS. A problem with the same formal structure is the initial-boundary-value problem

$$(9.7.a) \quad \frac{\partial}{\partial t} \alpha(u) - \Delta u \ni f, \quad x \in G,$$

$$(9.7.b) \quad \frac{\partial}{\partial t} \beta(v) + \frac{\partial u}{\partial \nu} \ni g \quad \text{and}$$

$$(9.7.c) \quad \frac{\partial u}{\partial \nu} \in \gamma(v - u), \quad s \in \Gamma,$$

for $t > 0$ with initial values specified at $t = 0$ for $\alpha(u)$ and $\beta(v)$. At each $t > 0$, $u(t)$ is a function on the bounded domain G in $\mathbb{R}^m$ with smooth boundary $\Gamma = \partial G$, and $v(t)$ is a function on Γ . Each of $\alpha(\cdot), \beta(\cdot), \gamma(\cdot)$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$. Thus, the system (9.7) consists of a generalized porous medium equation in the interior of G subject to a nonlinear dynamic constraint on the boundary Γ .

A remarkable variety of boundary conditions are included in (9.7). For example, if $\beta \equiv 0$ we have an explicit *Neumann boundary condition*, and if $\gamma \equiv 0$ it is homogeneous. If $\beta^{-1} = 0$, then $v \equiv 0$ and we have a nonlinear Robin constraint, and if $\gamma^{-1} = 0$ we get $v = u$ on Γ , and this satisfies the nonlinear *dynamic boundary condition*

$$\frac{\partial}{\partial t} \beta(u) + \frac{\partial u}{\partial \nu} \ni g \quad \text{in } L^1(\Gamma).$$

If $\beta^{-1} = 0$ and $\gamma^{-1} = 0$ we have the homogeneous *Dirichlet boundary condition* $u = 0$. Additionally we have seen in the previous Example 9.C that (9.7.b) and (9.7.c) can represent *boundary hysteresis*. That is, the map $u \mapsto \beta(v) \mapsto \frac{\partial u}{\partial \nu}$ is a hysteresis functional. Adsorption in porous media may be governed by conditions on the surfaces of the solid material that are of hysteresis type. If one assumes that the process is governed by certain thresholds, the adsorption rate shows a hysteresis phenomenon of the kind discussed above.

We can show that the problem (9.7) can be realized in the form (9.1) on the Banach space $L^1(G) \times L^1(\Gamma)$. In order to illustrate the types of estimates that are involved for the problem (9.7), we shall do this for simplicity in the special case of single valued functions, $\alpha(\cdot), \beta(\cdot), \gamma(\cdot)$. The operator $\mathbb{A}$ is constructed so that the

resolvent equation, $(I + \varepsilon \mathbb{A})([U, V]) \ni [f, g]$ with $\varepsilon > 0$, takes the form

$$\begin{aligned} U &= \alpha(u), \quad U - \varepsilon \Delta u \ni f \quad \text{in } L^1(G), \\ V &= \beta(v), \quad V + \varepsilon \frac{\partial u}{\partial \nu} \ni g, \quad \frac{\partial u}{\partial \nu} \in \gamma(v - u), \quad \text{in } L^1(\Gamma). \end{aligned}$$

In order to show how one obtains the essential estimates that are needed, multiply the respective equations by appropriate functions φ on G and ψ on Γ and integrate to obtain

$$\int_G (\alpha(u)\varphi + \varepsilon \vec{\nabla} u \cdot \vec{\nabla} \varphi) dx + \int_\Gamma (\beta(v)\psi + \varepsilon \gamma(v - u)(\psi - \varphi)) ds = \int_G f\varphi dx + \int_\Gamma g\psi ds.$$

This suggests the appropriate variational formulation and leads to the essential a-priori estimates. For example, if we choose $\varphi = \text{sgn}(u)$, $\psi = \text{sgn}(v)$ and can obtain simultaneously $\varphi = \text{sgn}(\alpha(u))$, $\psi = \text{sgn}(\beta(v))$ as before, then we obtain the stability estimate

$$\|\alpha(u)\|_{L^1(G)} + \|\beta(v)\|_{L^1(\Gamma)} \leq \|f\|_{L^1(G)} + \|g\|_{L^1(\Gamma)}.$$

By estimating similarly the *differences* of solutions, we establish that the *resolvent* map $[f, g] \mapsto [u, v] \mapsto [\alpha(u), \beta(v)]$ is a contraction, and this is the *accretiveness* of the operator $\mathbb{A}$. Under quite general conditions on the monotone graphs $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$, we find that $\mathbb{A}$ is m-accretive as desired. Our examples above show that it is worthwhile to retain as much generality as possible. As before, the Dirichlet operator $-\Delta$ in (9.4) can be replaced by a first-order operator as in Example 9.A in order to include scalar conservation laws with unilateral constraints or hysteresis functionals.

Appendix

A.1 Heat Conduction

Consider the diffusion of heat energy through a medium G in $\mathbb{R}^m$. For simplicity we shall assume that the medium is isotropic: its properties are independent of direction. The first experimental observation is that the heat energy in any subregion $G_0 \subset G$ is proportional to the mass of G_0 and its temperature. If we denote by $\rho(x)$ the density and by $u(x, t)$ the temperature at $x \in G$ and the time $t > 0$, then this assumption means that the heat energy in G_0 due to its temperature is given by $\int_{G_0} \rho(x)c(x)u(x, t) dx$, and this linear relationship defines the *heat capacity* or *specific heat* $c(x)$ of this material. Thus, $c(x)$ is the ratio of an increment in the heat energy of a unit mass to a corresponding increment in temperature at the point x . This assumption is good if the variations of temperature are not excessively large.

Fourier's law for heat conduction states that the rate at which heat energy flows through a surface element S is proportional to the surface area and to the temperature gradient in the direction normal to that surface. Thus, if we let $\vec{n}$ denote the unit normal on the surface S we have the total heat flow rate

$$- \int_S k(x) \vec{\nabla} u(x, t) \cdot \vec{n} ds$$

across S in the direction of $\vec{n}$. This defines the *conductivity*, $k(x)$; the *heat flux* is just $\vec{J}(x, t) = -k(x) \vec{\nabla} u(x, t)$, the vector surface density rate of heat flow. If the material is non-isotropic, then a temperature gradient will induce a heat flux which is not necessarily parallel; in that case conductivity is given as an $n \times n$ matrix K by $\vec{J}(x, t) = -K(x) \vec{\nabla} u(x, t)$. Thus, a unit temperature gradient $\vec{e}_j$ in the j direction induces the heat flow $-(K_{1,j}, K_{2,j}, \dots, K_{j,j})$. The matrix K is positive-definite and symmetric.

The *conservation of energy* shows that we have

$$\frac{\partial}{\partial t} \int_{G_0} \rho(x)c(x)u(x, t) dx = \int_{\partial G_0} k(x) \vec{\nabla} u(x, t) \cdot \vec{n} + \int_{G_0} f(x, t) dx$$

for every $G_0 \subset G$ and $t > 0$, where $f(x, t)$ denotes the volume-distributed heat sources in G and $\vec{n}$ is the unit outward normal. From Gauss' theorem it follows that

$$\int_{G_0} \frac{\partial}{\partial t} (\rho(x)c(x)u(x, t)) dx = \int_{G_0} \vec{\nabla} \cdot k(x) \vec{\nabla} u(x, t) dx + \int_{G_0} f(x, t) dx$$

for every such $G_0 \subset G$, so we obtain the linear *heat equation*

$$(1.1.a) \quad \frac{\partial}{\partial t} (\rho(x)c(x)u(x, t)) - \vec{\nabla} \cdot k(x) \vec{\nabla} u(x, t) = f(x, t) \text{ in } G.$$

Surface Discontinuity. The above accounts for energy balance when first-order derivatives of the temperature are continuous in the interior of G . Now let S be an $(n-1)$ -dimensional surface in G , and suppose we have a solution which is smooth except along S . Note that since temperature is obtained as an average over some small generic neighborhood, it is always expected to be continuous. However, $\vec{\nabla}u$ is not necessarily continuous. In fact, it is the *flux* $k(x)\vec{\nabla}u(x,t) \cdot \vec{n} = k(x)\frac{\partial u}{\partial n}$ across the surface S with normal $\vec{n}$ which we expect to be continuous. These two observations yield the *interface conditions*

$$u(x_+) = u(x_-), \quad k(x_+)\frac{\partial u}{\partial n}(x_+) = k(x_-)\frac{\partial u}{\partial n}(x_-)$$

where x_+ and x_- denote the limiting values from the two sides at $x \in S$. It follows that the directional derivative $\frac{\partial u}{\partial n}$ is continuous exactly when $k(x)$ is continuous. In particular, if we have a non-homogeneous medium in which the material coefficient $k(x)$ is discontinuous across a surface, S , then the temperature $u(x,t)$ is continuous across S but the derivative $\frac{\partial u}{\partial n}$ will necessarily have a jump.

Boundary Conditions. Next we consider the conditions on the boundary of the region. One could measure the temperature on a portion Γ_0 of the boundary, $\Gamma \equiv \partial G$. Then we have the *Dirichlet boundary condition*

$$u(s,t) = h(s,t), \quad s \in \Gamma_0,$$

where $h(s,t)$ is known. One could likewise measure the heat flow across a part Γ_1 of Γ , so we obtain the *Neumann boundary condition*

$$k(s)\vec{\nabla}u(s,t) \cdot \vec{n} = g(s,t), \quad s \in \Gamma_1.$$

This condition also describes the situation in which there is a heat source of density $g(x,t)$ distributed along Γ_1 . More generally, one can assume the heat flux on the boundary is also driven by the difference between inside and outside temperatures,

$$(1.1.b) \quad k(s)\vec{\nabla}u(s,t) \cdot \vec{n} + k_0(s)(u(s,t) - h(s,t)) = g(s,t), \quad s \in \Gamma_1.$$

This is called a *Robin boundary condition*, and it corresponds to *Newton's law* of cooling on the surface: the heat loss rate is proportional to the temperature difference across the surface. It is a *discrete* Fourier law in which $k_0(s)$ is the surface-distributed conductivity. For example, if the boundary is insulated along Γ_1 , then we have $k_0(s) = 0$. Note that the first two types of boundary condition are obtained formally as the special cases $k_0 \rightarrow +\infty$ and $k_0 \rightarrow 0$, respectively, and the third type of boundary condition results from intermediate values for $k_0(s)$.

A problem with a similar formal structure arises when we permit the boundary to consist of a material with the specific heat given by $c_0(\cdot)$. If we assume that the heat energy is supplied to the boundary by the flux from the interior, in addition to that from a source distribution on the boundary, F_0 , we are led to the *dynamic boundary condition*

$$(1.1.b') \quad \frac{\partial}{\partial t} \rho_0(s)c_0(s)u(s,t) + k(s)\vec{\nabla}u(s,t) \cdot \vec{n} \ni F_0(s,t), \quad s \in \Gamma,$$

for $t > 0$. If we also consider the local diffusion in the tangential coordinates of the boundary, then we obtain the corresponding elliptic Laplace-Beltrami operator in (1.1.b') for the manifold Γ . The appropriate initial-boundary-value problem in

each case is to seek a solution of the heat equation (1.1.a) subject to a boundary condition (1.1.b) or (1.1.b') and an initial condition

$$(1.1.c) \quad \rho(x)c(x)u(x,0) = \rho(x)c(x)u_0(x), \quad x \in G,$$

and, additionally, in the case of (1.1.b'),

$$(1.1.c') \quad \rho_0(s)c_0(s)u(s,0) = \rho_0(s)c_0(s)u_0(s), \quad s \in \Gamma,$$

hence, the initial energy is specified.

Nonlinear Models.

Here we assume that the density is constant, and we normalize it to unity by an appropriate change of variable. If large variations in the temperature are considered, then it may be appropriate to permit the material properties to be temperature dependent. That is, we could consider *semilinear* equations of the form

$$(1.2) \quad \frac{\partial}{\partial t}c(u) - \Delta k(u) = f(x, t)$$

for which we have $\frac{\partial}{\partial t}c(u) = c'(u)\frac{\partial u}{\partial t}$, where $c'(u)$ denotes the specific heat at the temperature u , and $-\Delta k(u) = -\vec{\nabla} \cdot k'(u)\vec{\nabla}u$, so $k'(u)$ is the corresponding conductivity. Since the specific heat and conductivity are positive quantities, it follows that the corresponding functions, $c(\cdot)$ and $k(\cdot)$, are both *monotone*. By means of a change of variable, we can eliminate either one of $c(\cdot)$ or $k(\cdot)$ with no loss of generality. The resulting equation with either of $c(\cdot)$ or $k(\cdot)$ is called the *porous medium equation*. Nonlinear boundary conditions could also be considered. Relevant examples include those of the form

$$(1.3) \quad \frac{\partial}{\partial t}c_0(u) + \vec{\nabla}k(u(s, t)) \cdot \vec{n} + k_0(u(s, t) - h(s, t)) = g(x, t), \quad s \in \Gamma$$

in which $k_0(\cdot)$ and $c_0(\cdot)$ are also monotone functions. Equations of the form (1.2) arise in a variety of applications, and it is common to have *singular* or *degenerate* situations in which $c'(u) = 0$ or $k'(u) = 0$ for values of u in some interval. Next we discuss an example in which monotone but multi-valued *graphs* or *relations* appear. These are dual to the preceding situation, since portions of the graphs are vertical rather than horizontal.

The Stefan Problem. We consider a model of heat diffusion in the situation where there is a *phase change* arising from the freezing or thawing of the medium at a fixed temperature. We begin with the description of the phase relation between energy and temperature. Consider a unit volume of ice at temperature $u < 0$ and apply a uniform heat source of intensity F , so $e \equiv Ft$ is the accumulated *internal energy* after t units of time. The temperature increases according to the monotone function $e = c(u)$ until it reaches the value $u = 0$; the positive function $c'(u)$ is the *specific heat*. Then the temperature remains at $u = 0$ until L units of additional heat have been added; $L > 0$ is the *latent heat*. During this period there is a fraction w of water coexisting with the ice, and w increases at the constant rate F/L . The *water fraction* w , $0 \leq w \leq 1$, is the phase variable. After all the ice has melted, $w = 1$ and the temperature u begins to rise again according to $e = c(u) + L$. If the process is reversed by drawing heat out of the unit volume, the temperature falls according to $e = c(u) + L$ until it reaches $u = 0$, then w decreases until it reaches

$w = 0$, and thereafter the temperature u falls with $e = c(u)$. Note that the freezing and the melting took place at $u = 0$. This is the situation which leads to the traditional *Stefan problem*. The relation between energy and temperature is given in terms of the Heaviside graph by $e \in c(u) + LH(u)$. (Recall that the *Heaviside graph* $H(\cdot)$ is defined by $H(u) = \{0\}$ if $u < 0$, $H(0) = [0, 1]$, and $H(u) = \{1\}$ if $u > 0$.) The energy is then given in the form $e = c(u) + Lw$, with $w \in H(u)$.

We shall formulate a free-boundary problem which describes heat conduction through a domain G in Euclidean space R^m subject to the constitutive assumptions above on the phase relation between energy and temperature. This is called the *Stefan problem*. Denote the boundary of G by ∂G and set $\Omega = G \times (0, \infty)$. The temperature at the point $x \in G$ and the time $t > 0$ is $u(x, t)$ and the smooth monotone functions $c(u)$, $k(u)$ are given with $c(0) = k(0) = 0$; their derivatives $c'(u)$, $k'(u)$ denote the *specific heat* and *conductivity*, respectively, of ice-water at temperature u . The phase change between water and ice occurs at $u = 0$. The space-time region Ω is then separated into an ice region Ω_- where $u < 0$, a water region Ω_+ where $u > 0$, and a region Ω_0 where $u = 0$ and in which the phase depends on its preceding history.

Let $w(x, t)$ be the fraction of water at $(x, t) \in \Omega$, and note that according to our constitutive assumptions above we have $w \in H(u)$, the Heaviside graph. The energy is given by $e = c(u) + Lw$. Let S_- be the boundary of Ω_- in Ω and S_+ the boundary of Ω_+ in Ω . The *unit normal* $N = (N_1, \dots, N_m, N_t)$ on $S_- \cup S_+$ is oriented out of Ω_- and into Ω_+ , and hence it is consistent on $S_- \cap S_+$. We shall denote by $[g]$ the *saltus* or jump in values of the function g across the boundaries, S_- and S_+ , in the direction of N : for $(x, t) \in S_- \cup S_+$

$$[g(x, t)] = \lim_{h \rightarrow 0^+} \left\{ g((x, t) + hN) - g((x, t) - hN) \right\}.$$

The strong form of the *Stefan problem* is to find a pair of functions u and w on Ω for which

$$(1.4.a) \quad \frac{\partial}{\partial t} c(u) - \Delta k(u) = F(x, t) \text{ in } \Omega_- \cup \Omega_+,$$

$$(1.4.b) \quad \frac{\partial}{\partial t} Lw = F(x, t) \text{ in } \Omega_0, \quad w \in H(u) \text{ in } \Omega,$$

$$(1.4.c) \quad [\nabla k(u)] \cdot (N_1, \dots, N_m) = LN_t[w] \text{ on } S_- \cup S_+,$$

$$(1.5.a) \quad u(x, 0) = u_0, \quad x \in G,$$

$$(1.5.b) \quad w(x, 0) = w_0(x) \in [0, 1], \text{ where } u_0(x) = 0,$$

$$(1.6) \quad u(s, t) = 0, \quad s \in \partial G, \quad t > 0,$$

where a distributed source $F \in L^1(\Omega)$ is given in addition to $c(\cdot)$, $k(\cdot)$, L , u_0 , w_0 . The classical (possibly nonlinear) heat equation (1.4.a) determines the temperature where $u \neq 0$. The water fraction is given by the relation (1.4.b) which was described above, so we have $w = 0$ in the ice, Ω_- , $w = 1$ in the water, Ω_+ , and w is between 0 or 1 in the *mushy region*, Ω_0 . Let n be the unit vector in the direction $(N_1, \dots, N_m)$, and let V be the *velocity* of S_- or S_+ at time t in the direction of n . By dividing (1.4.c) by $(N_1^2 + \dots + N_m^2)^{1/2}$, we obtain

$$\left[\frac{\partial}{\partial n} k(u) \right] + LV[w] = 0 \text{ on } S_- \cup S_+,$$

and this is equivalent to (1.4.c). It means the difference in heat flux across the *free boundary* S_+ determines the velocity V of that boundary by melting the fraction of ice $1 - w = [w]$ with latent heat L , and similarly the velocity of S_- is determined by the freezing of the fraction of water $w = [w]$. The Dirichlet boundary condition (1.6) is used here for simplicity, but any of the usual types can just as easily be attained.

In order to obtain a *weak formulation* of the Stefan problem, we consider a solution u, w of (1.4) for which u is smooth in each of Ω_- , Ω_0 , Ω_+ , and the discontinuities in flux supply the energy to drive the surfaces S_- and S_+ according to (4.c). We begin by computing $\frac{\partial e}{\partial t} - \Delta k(u)$ in the sense of distributions on Ω . Thus, for $\varphi \in C_0^\infty(\Omega)$ we have

$$\left\langle \frac{\partial e}{\partial t} - \Delta k(u), \varphi \right\rangle \equiv \int_{\Omega} \left\{ -(c(u) + Lw)\varphi_t - k(u)\Delta\varphi \right\}.$$

Since $k(u)$ is smooth in each region, we have

$$\begin{aligned} & \int_{\Omega} \left\{ -(c(u) + Lw)\varphi_t + \nabla k(u) \cdot \nabla \varphi \right\} \\ &= \int_{\Omega_-} \left\{ -c(u)\varphi_t + \nabla k(u) \cdot \nabla \varphi \right\} + \int_{\Omega_0} \left\{ -(Lw)\varphi_t \right\} \\ & \quad + \int_{\Omega_+} \left\{ -(c(u) + L)\varphi_t + \nabla k(u) \cdot \nabla \varphi \right\} \end{aligned}$$

From Gauss' theorem we can write these three successive integrals in the respective forms

$$\begin{aligned} & \int_{\Omega_-} (c(u)_t - \Delta k(u)) \varphi + \int_{S_-} \{ \nabla k(u) \cdot (N_1, \dots, N_m) \} \varphi, \\ & \int_{\Omega_0} (Lw)_t \varphi + \int_{S_-} \{ (Lw) N_t \} \varphi - \int_{S_+} \{ (Lw) N_t \} \varphi, \text{ and} \\ & \int_{\Omega_+} (c(u)_t - \Delta k(u)) \varphi - \int_{S_+} \{ \nabla k(u) \cdot (N_1, \dots, N_m) - L N_t \} \varphi. \end{aligned}$$

By adding these we obtain

$$\begin{aligned} \left\langle \frac{\partial e}{\partial t} - \Delta k(u), \varphi \right\rangle &= \int_{\Omega_- \cup \Omega_0 \cup \Omega_+} ((c(u) + Lw)_t - \Delta k(u)) \varphi \\ & \quad + \int_{S_- \cup S_+} (-[\nabla k(u)] \cdot (N_1, \dots, N_m) + L[w] N_t) \varphi. \end{aligned}$$

Hence, it follows that (1.4) is equivalent to

$$(1.7) \quad \frac{\partial}{\partial t} (c(u) + Lw) - \Delta k(u) = F(x, t), \quad w \in H(u),$$

where w, u are appropriately smooth. We have shown that the weak form of the Stefan problem is to find a pair $u \in L^1(\Omega)$, $w \in L^\infty(\Omega)$ for which (1.7) and (1.5) hold, and $k(u) \in L^2(0, T; H_0^1(G))$. This last condition implies (1.6) if $k'(0) > 0$.

We close with some remarks on the enthalpy functional (1.4.b). The simple relation $H(u)$ described above is an idealization. Specifically, during the phase change the temperature u does not necessarily remain exactly constant but may

increase at a very small rate. Thus, it is reasonable to replace the Heaviside relation by a (single-valued) monotone function which closely approximates it. Also, if one could manage to force temperatures past the phase-change temperature, the phase w would not be expected to respond instantly, but only at a very high rate. We can easily accommodate both of these modifications, and the latter lead to various *dynamic models* of phase change.

Hysteresis Models. The use of multi-valued graphs permits a very elegant treatment of a class of parabolic problems with *hysteresis*, and we shall describe an example of the form

$$(1.8) \quad \frac{\partial}{\partial t} (a(u) + \mathcal{H}(u)) - \Delta u = f$$

in which $\mathcal{H}$ denotes a *hysteresis* functional, that is, its value depends not only on the current value of the input, u , but also on the *history* of the input in a very nonlinear way.

We first consider an elementary but fundamental example of hysteresis. It depends on three parameters, α , β , and ϵ , with $0 < \epsilon$, $\alpha < \beta$. Denote by $[x]_+$ and $[x]_-$, respectively, the positive and negative parts of the real number x . Let's look at the response $w(\cdot)$ that arises from a given input, $u(\cdot)$. The output $w = \mathcal{H}_{\alpha, \beta, \epsilon}(u)$ varies according to the following: if $u > \beta + \epsilon$, then $w = 1$; if $u < \alpha$, then $w = 0$; if $\alpha < u < \beta + \epsilon$, then $0 \leq w \leq 1$ and

$$w'(t) = \begin{cases} \left[\frac{u'(t)}{\epsilon} \right]_+ & \text{if } w = \frac{u - \beta}{\epsilon}, \\ 0 & \text{if } \frac{u - \beta}{\epsilon} < w < \frac{u - \alpha}{\epsilon}, \\ \left[\frac{u'(t)}{\epsilon} \right]_- & \text{if } w = \frac{u - \alpha}{\epsilon}. \end{cases}$$

For example, suppose we start with $u(0) = 0$ and $w(0) = 0$. As long as $u(t)$ remains between α and β , the output w stays constant at the value $w(t) = 0$. If $u(t)$ increases to β and then beyond $\beta + \epsilon$, then $w(t)$ increases to $+1$ where it remains until $u(t)$ gets down to $\alpha + \epsilon$. If $u(t)$ decreases below α , then $w(t)$ will drop to 0 and remain there until $u(t)$ again reaches β , and so on. It is clear that the output $w(t)$ depends not only on the present value but also on the history of the input $u(s)$ for previous times, $0 < s < t$. The limiting case obtained from $\epsilon \rightarrow 0$ is the *relay* hysteresis functional $\mathcal{H}_{\alpha, \beta}$. When $\alpha = \beta = 0$ this reduces further to the *Heaviside graph*.

In order to see how an ordinary differential equation with constraint can produce such a hysteresis functional, we consider a somewhat more general situation. Let a maximal monotone graph $b(\cdot)$ be given; this hysteresis model will be of the type *generalized play* described by horizontal translates of $w \in b(u)$. The hysteresis functional $\mathcal{H}_{\alpha, \beta, \epsilon}$ described above is given by the choice $b = H_\epsilon$, where

$$H_\epsilon(r) = \begin{cases} 1 & \text{if } r \geq \epsilon \\ \frac{r}{\epsilon} & \text{if } 0 < r < \epsilon \\ 0 & \text{if } r \leq 0. \end{cases}$$

Introduce a new variable, v , to represent the *phase* constraints:

$$w \in b(v), \quad u - \beta \leq v \leq u - \alpha.$$

We use the *sign function* (or graph) sgn to realize these constraints. Recall that it is defined by $\text{sgn}(x) = 1$ if $x > 0$, $\text{sgn}(x) = -1$ if $x < 0$, and $\text{sgn}(0) = [-1, 1]$. We shall use its scaled version, $\text{sgn}_{\alpha,\beta}$, defined by $\text{sgn}_{\alpha,\beta}(x) = \beta$ if $x > 0$, $\text{sgn}_{\alpha,\beta}(x) = \alpha$ if $x < 0$, and $\text{sgn}_{\alpha,\beta}(0) = [\alpha, \beta]$. Thus, let $u(t)$ be a time-dependent input to this generalized play model, and let $w(t)$ be the corresponding output or response. There is at each time t a corresponding phase variable $v(t)$ which is related to $w(t)$ and $u(t)$ as above, and so it is required that $w(t)$ be non-decreasing when $v(t) = u(t) - \beta$, non-increasing when $v(t) = u(t) - \alpha$, and stationary ($w'(t) = 0$) in the interior region where $u - \beta < v < u - \alpha$. This is equivalent to requiring that $w(t), v(t)$ satisfy

$$w(t) \in b(v(t)) \quad , \quad w' + \text{sgn}_{-\beta, -\alpha}^{-1}(v(t) - u(t)) \ni 0.$$

Thus, we are led to ordinary differential equations of the form

$$w(t) \in b(v(t)) \quad , \quad w'(t) + c(v(t) - u(t)) \ni 0,$$

with maximal monotone graphs $b(\cdot)$ and $c(\cdot)$, as models of hysteresis in which the output is the solution $w(t)$ with input $u(t)$.

We have described above the *relay hysteresis* functional which is determined by the sign graph and the two parameters $\alpha < \beta$. These translation parameters are the switching positions of the relay, and we denote the output of this relay by $w = \mathcal{H}_{\alpha,\beta}(u)$. This operator can be represented by a rectangular loop which is the input-output graph of the relay which switches up to the value $w = 1$ at $u = \beta$ and down to the value $w = -1$ at $u = \alpha$. When this hysteresis functional is combined with the nonlinear diffusion equation as above, we have a system of the form

$$(1.9.a) \quad \frac{\partial}{\partial t} a(u(x, t)) - \Delta u(x, t) - c(v(x, t) - u(x, t)) \ni f(x, t), \quad x \in G, \quad t \in (0, T],$$

$$(1.9.b) \quad \frac{\partial}{\partial t} b(v(x, t)) + c(v(x, t) - u(x, t)) \ni g(x, t),$$

$$(1.9.c) \quad -\frac{\partial}{\partial \nu} u(s, t) \in d(u(s, t)), \quad s \in \partial\Omega,$$

with $b(\cdot) = H(\cdot)$, $c(\cdot) = \text{sgn}_{-\beta, -\alpha}^{-1}(\cdot)$ and $g(x, t) = 0$. This is a degenerate parabolic system of coupled equations with Neumann type boundary conditions for which the initial conditions $a(u(x, 0))$ and $b(v(x, y, 0))$ are to be specified, and it realizes the parabolic equation (1.8) with the indicated relay hysteresis. We can show that the dynamics of problem (1.9) is determined by a nonlinear semigroup of contractions on the Banach space $L^1(G) \times L^1(G)$.

The Super-Stefan Problem. We extend the Stefan problem to include hysteresis effects that result from the assumption that the melting and the freezing temperatures are *different*. The ice melts at a slightly positive temperature and freezing occurs after a slightly negative temperature is reached.

We begin with the description of the phase relation between energy and temperature. Let α and β be given numbers with $\alpha < 0 < \beta$. Begin with a unit volume of ice at temperature $u < \alpha$ and apply the heat source F . The temperature increases according to the relation $e = c(u)$ until it reaches the value $u = \beta > 0$

where it remains until L units of additional heat have been added. During this period the fraction w of water increases at the rate F/L . After all the ice has melted, $w = 1$ and the temperature u begins to rise again according to $e = c(u) + L$. If the process is reversed, the temperature falls according to $e = c(u) + L$ until it reaches $u = \alpha < 0$, then w decreases until it reaches $w = 0$, and thereafter the temperature u falls with $e = c(u)$. Note that the freezing took place at $u = \alpha$ and the melting at $u = \beta$. If $\alpha = \beta$ this is just the traditional Stefan problem. However this model permits superheated ice and supercooled water, and it is here that hysteresis occurs.

We formulate the corresponding free-boundary problem, the *Super-Stefan problem*. The phase change from water to ice occurs at $u = \alpha < 0$ and from ice to water at $u = \beta > 0$. The space-time region Ω is then separated into an always-ice region Ω_- where $u < \alpha$, an always-water region Ω_+ where $u > \beta$, and a region Ω_0 where $\alpha \leq u \leq \beta$ and in which the phase depends on its preceding history. According to our constitutive assumptions, the water fraction satisfies $w \in \mathcal{H}_{\alpha,\beta}(u)$, and the energy is given by $e = c(u) + Lw$. Let S_- be the boundary of Ω_- in Ω and S_+ the boundary of Ω_+ in Ω . The *unit normal* $N = (N_1, \dots, N_m, N_t)$ on $S_- \cup S_+$ is oriented out of Ω_- and Ω_+ , and hence into Ω_0 . The strong form of the problem is to find a pair of functions u and w on Ω for which

$$(1.10.a) \quad \frac{\partial}{\partial t} c(u) - \Delta k(u) = 0 \text{ in } \Omega_- \cup \Omega_0 \cup \Omega_+ ,$$

$$(1.10.b) \quad w \in \mathcal{H}_{\alpha,\beta}(u) \text{ in } \Omega ,$$

$$(1.10.c) \quad [\nabla k(u)] \cdot (N_1, \dots, N_m) = LN_t[w] \text{ on } S_- \cup S_+ ,$$

$$(1.11.a) \quad u(x, 0) = u_0 , \quad x \in G ,$$

$$(1.11.b) \quad w(x, 0) = w_0(x) \in [0, 1] , \text{ where } \alpha \leq u_0(x) \leq \beta ,$$

$$(1.12) \quad u(s, t) = 0 , \quad s \in \partial G , \quad t > 0 .$$

One can show as before that the *weak form* of the Super-Stefan problem is to find a pair $u \in L^1(\Omega)$, $w \in L^\infty(\Omega)$ for which

$$(1.13) \quad \frac{\partial}{\partial t} (c(u) + Lw) - \Delta k(u) = 0 , \quad w \in \mathcal{H}_{\alpha,\beta}(u) ,$$

and $k(u) \in L^2(0, T; H_0^1(G))$. With the appropriate change of variable, this is just the equation (1.8) with the simple relay hysteresis.

Boundary Evolution System. Next we describe a problem with the same formal structure as (1.9), a degenerate-parabolic initial boundary value problem for $t > 0$

$$(1.14.a) \quad \frac{\partial}{\partial t} a(u) - \Delta u \ni f , \quad x \in G ,$$

$$(1.14.b) \quad \frac{\partial}{\partial t} b(v) + \frac{\partial u}{\partial \nu} \ni g \text{ and}$$

$$(1.14.c) \quad \frac{\partial u}{\partial \nu} \in c(v - u) , \quad s \in \Gamma$$

with initial values specified at $t = 0$ for $a(u)$ and $b(v)$. At each $t > 0$, $u(t)$ is a function on the bounded domain G in $\mathbb{R}^m$ with smooth boundary Γ , and $v(t)$ is a function on Γ . Each of $a(\cdot)$, $b(\cdot)$, $c(\cdot)$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$. Thus,

the system (1.14) consists of a generalized porous medium equation in the interior of G subject to a nonlinear dynamic constraint on the boundary.

A variety of boundary conditions is obtained in (1.14). For example, if $b \equiv 0$ we have an explicit *Neumann boundary condition*, and if $c \equiv 0$ it is homogeneous. If $b(0) = \mathbb{R}$ (i.e., $b^{-1} = 0$), then $v \equiv 0$ and we have a nonlinear Robin constraint, and if $c(0) = \mathbb{R}$ we get $v = u$ on Γ , and this satisfies the nonlinear *dynamic boundary condition*

$$(1.14.b') \quad \frac{\partial}{\partial t} b(u) + \frac{\partial u}{\partial \nu} \ni g.$$

If $b(0) = c(0) = \mathbb{R}$ we have the homogeneous *Dirichlet boundary condition*. Note that (1.14.b) together with (1.14.c) can represent *boundary hysteresis*. The dynamics of the problem (14) is given by a nonlinear semigroup of contractions on the Banach space $L^1(G) \times L^1(\Gamma)$.

Composite Media.

We shall develop some models for the flow of heat in a *composite material* composed of two finely interwoven and (possibly) connected components. Note that it is impossible to satisfy both of these geometric constraints in $\mathbb{R}^2$. We treat the two components of the system as independent media with different physical parameters. The discontinuities in these parameter values across the common interface can be severe, with the ratios of their values in the two media being of some orders of magnitude. Consequently, the exact microscopic model, written as a classical interface problem, is numerically and analytically intractable. Thus one constructs models by averaging methods. We shall first construct the simplest such type of model for conduction, a *parallel flow* model. At each point of the region, we center a generic neighborhood, and then we compute two quantities at that point, the average temperature of the first material within that neighborhood and then the average temperature of the second material within that same neighborhood. Thus there are two temperatures identified with each point of the region, and each of these two temperature fields satisfies the classical heat conduction equation together with an appropriate coupling term that combines them into a system.

Consider now small temperature variations in a system consisting of two components as above. We let $u(x, t)$ denote the temperature of the first material (averaged over the neighborhood centered at x), and let $v(x, t)$ denote the corresponding temperature in the second material at x . In order to account for the exchange of heat between the two components, we assume the simplest (linear) coupling, namely, Newton's law: the heat flux between the components is proportional to the difference between their temperatures. This leads to the linear system

$$(1.15) \quad \begin{aligned} \frac{\partial}{\partial t} \rho_1(x) c_1(x) u(x, t) - \vec{\nabla} \cdot k_1(x) \vec{\nabla} u(x, t) + k_0(u(x, t) - v(x, t)) &= f_1(x, t), \\ \frac{\partial}{\partial t} \rho_2(x) c_2(x) v(x, t) - \vec{\nabla} \cdot k_2(x) \vec{\nabla} v(x, t) + k_0(v(x, t) - u(x, t)) &= f_2(x, t), \end{aligned}$$

the simplest example of a *parallel flow model* which describes heat conduction in a composition of two media. The subscripts refer to the characteristic properties of the respective media in the two components, and k_0 is a conductivity for the volume distributed heat exchange between the components. The name *parallel flow* is used because this is the same system that arises from the description of heat flow in two parallel planes that are in contact.

Fissured Structures. We consider the modifications of the system (1.15) that are appropriate to model a fissured medium. The characteristics of a fissured structure are that the first component is a matrix which occupies a much larger volume than the fissure system which comprises the second component and that the matrix is much more resistant to heat conduction than is the conducting material which is imbedded in the pores of the matrix and more highly concentrated in the fissure system. As a consequence, most of the heat is conducted through the system of fissures, while storage of heat takes place primarily inside the matrix. Some part of the heat flow is through the matrix, but the primary flow is that from the matrix into the fissures followed by flow within the fissures. There will be some flow in the matrix: the temperatures at nearby parts of the matrix can influence each other directly, not just indirectly via the system of fissures. This effect can be substantially less prominent than the bulk flow of heat through the fractures, but it can have a noticeable effect when the connectivity of the matrix is sufficiently well developed or the flow through the fissure system is not so overwhelming. Suppose the composition is such that the first material occurs as a system of particles suspended within the second in such a way that they are isolated from one another. Then there is no direct diffusion from one of these particles to the neighboring particles, but only an exchange into the surrounding second material. We realize this situation in the system (1.15) by setting $k_1 = 0$. The resulting system consists of an ordinary differential equation coupled to a parabolic partial differential equation,

$$(1.16.a) \quad \frac{\partial}{\partial t} \rho_1(x) c_1(x) u(x, t) + k_0(u(x, t) - v(x, t)) = f_1(x, t) ,$$

$$(1.16.b) \quad \frac{\partial}{\partial t} \rho_2(x) c_2(x) v(x, t) - \vec{\nabla} \cdot k_2(x) \vec{\nabla} v(x, t) + k_0(v(x, t) - u(x, t)) = f_2(x, t) ,$$

and this is called a *first-order kinetic model*. The equation (16.a) serves as a memory term. In the model of heat flow in two parallel planes, the system (1.16) results from the assumption that the top layer consists of a fine distribution of isolated particles.

If we additionally assume that the second material is a connected component so thinly distributed throughout the composite that its volume is negligible, then this second component cannot store any substantial amount of heat energy, and we realize this assumption by setting $\rho_2 = 0$ to obtain

$$\begin{aligned} \frac{\partial}{\partial t} \rho_1(x) c_1(x) u(x, t) + k_0(u(x, t) - v(x, t)) &= f_1(x, t) , \\ -\vec{\nabla} \cdot k_2(x) \vec{\nabla} v(x, t) + k_0(v(x, t) - u(x, t)) &= f_2(x, t) . \end{aligned}$$

This is a coupled ordinary-elliptic system in which all storage is in the first component and all diffusion is in the second component. We can eliminate v to obtain the *fissured medium equation*

$$\begin{aligned} \frac{\partial}{\partial t} \rho_1(x) c_1(x) u(x, t) - \vec{\nabla} \cdot k_2(x) \vec{\nabla} (u(x, t) + \frac{1}{k_0} \frac{\partial}{\partial t} \rho_1(x) c_1(x) u(x, t)) \\ = (I - \frac{1}{k_0} \vec{\nabla} \cdot k_2(x) \vec{\nabla}) f_1(x, t) + f_2(x, t) . \end{aligned}$$

This can also be written in the form

$$\begin{aligned} & \left(I - \frac{1}{k_0} \vec{\nabla} \cdot k_2(x) \vec{\nabla}\right) \frac{\partial}{\partial t} \rho_1(x) c_1(x) u(x, t) - \vec{\nabla} \cdot k_2(x) \vec{\nabla} u(x, t) \\ &= \left(I - \frac{1}{k_0} \vec{\nabla} \cdot k_2(x) \vec{\nabla}\right) f_1(x, t) + f_2(x, t) \end{aligned}$$

which we recognize as the *Yosida approximation* with parameter $\alpha = \frac{1}{k_0}$ of the heat equation.

For the construction of parallel flow models of heat flow when the temperature variations are larger and consequently the components are described by the semilinear equations (1.2), we obtain the nonlinear system

$$(1.17) \quad \begin{aligned} \frac{\partial}{\partial t} a_1(u) - \Delta k_1(u) + \tau(u - v) &= f_1(x, t) \\ \frac{\partial}{\partial t} a_2(v) - \Delta k_2(v) - \tau(u - v) &= f_2(x, t) \end{aligned}$$

in which the coupling is described by the monotone function $\tau(\cdot)$, and the monotone functions $a_1(\cdot)$ and $a_2(\cdot)$ include the density. As a naturally occurring example of a *Stefan system*, one finds fissured media in which, for example, immobile water is the conductor and it can undergo a phase change. Then the first component is the water temperature in the matrix of porous material and the second component is water temperature in the system of fissures. Both will undergo a phase change, so one gets a Stefan problem in each component of the system (1.17). Furthermore, since each component acts on the other as a distributed source, there will always be mushy regions present. This system generalizes (1.9), and its dynamics is determined by a nonlinear semigroup of contractions on $L^1(G) \times L^1(G)$.

Distributed Microstructure Models. We describe an elementary example in which a *distributed microstructure* model arises very naturally. The problem concerns the following situation. A system of pipes is used to absorb heat from circulating hot water and then later to return this heat to colder water, thereby serving as an energy storage device. This system has two sources of singularity, a geometric one due to the high ratio between the pipe length and the cross section, and a material one due to the very different conduction properties of the two materials involved. We seek a model in which these two singularities are properly balanced in order to obtain a good description of the exchange process.

We begin by examining the heat exchange in a single pipe in such a system. The heat transport in the water is primarily convective, due to the unit velocity of the water. The transport in the walls of the pipe is purely conductive and relatively very slow. The cross-sections of the pipe are very small, and these must be properly scaled to accurately model the exchange of heat between them and the water. We introduce a small parameter $\varepsilon > 0$ to quantify this. A reference cell $Y = \{y = (y_1, y_2) \in \mathbb{R}^2 : |y_1|^2 + |y_2|^2 < 2^2\}$ defines the structure of the cross-sections, and we write the parts of this double component domain as $\bar{Y} = \bar{Y}_1 \cup \bar{Y}_2$, with $Y_1 = \{y \in \mathbb{R}^2 : \|y\| < 1\}$ representing the internal flow region and $Y_2 = \{y \in \mathbb{R}^2 : 1 < \|y\| < 2\}$ the walls of the pipe. The boundary of Y_1 is the unit circle, Γ_{11} , and that of Y is a larger circle, Γ_{22} . The boundary of Y_2 is given by $\Gamma_2 = \Gamma_{11} \cup \Gamma_{22}$. By ν we denote the unit vector normal to Γ_2

which points in the direction *out* of Y_2 . We shall represent the very small cross-sections by $Y_1^\varepsilon = \varepsilon Y_1 = \{z = (z_1, z_2) = \varepsilon(y_1, y_2) \in \mathbb{R}^2 : (y_1, y_2) \in Y_1\}$ and $Y_2^\varepsilon = \varepsilon Y_2 = \{z = \varepsilon y \in \mathbb{R}^2 : y \in Y_2\}$. Similarly, we denote the corresponding boundaries by $\Gamma_{11}^\varepsilon, \Gamma_{22}^\varepsilon$, and Γ_2^ε .

Let the interval $G = (0, 1)$ denote the axis of the very narrow pipe in which water is flowing at unit velocity through the cross section Y_1^ε and for which Y_2^ε is the pipe wall, a material of relatively *very low* conductivity. This situation is described by the initial-boundary-value problem

$$\begin{aligned} \frac{\partial U^\varepsilon}{\partial t} + \frac{\partial U^\varepsilon}{\partial x} &= \frac{\partial^2 U^\varepsilon}{\partial x^2} + \nabla_z \cdot \nabla_z U^\varepsilon(x, z, t), & z \in Y_1^\varepsilon, & x \in G, \\ \frac{\partial U^\varepsilon}{\partial t} &= \varepsilon^2 \left(\nabla_z \cdot \nabla_z U^\varepsilon(x, z, t) + \frac{\partial^2 U^\varepsilon}{\partial x^2} \right), & z \in Y_2^\varepsilon, & x \in G, \\ \nabla_z U^\varepsilon \cdot \nu|_{Y_1^\varepsilon} &= \varepsilon^2 \nabla_z U^\varepsilon \cdot \nu|_{Y_2^\varepsilon}, & z \in \Gamma_{11}^\varepsilon, & x \in G, \\ \nabla_z U^\varepsilon \cdot \nu &= 0, & z \in \Gamma_{22}^\varepsilon, & x \in G, \end{aligned}$$

which is the *exact ε -model*. The ε^2 -scaled conductivity permits *very high* gradients in $G \times Y_2^\varepsilon$. However, only those components in the cross-section direction are responsible for the exchange flux, and in order to see that it is balanced with the other transport terms we rescale with $z = \varepsilon y$ and $\nabla_z = \frac{1}{\varepsilon} \nabla_y$ to get

$$\begin{aligned} (1.18.a) \quad \frac{\partial U^\varepsilon}{\partial t} + \frac{\partial U^\varepsilon}{\partial x} &= \frac{\partial^2 U^\varepsilon}{\partial x^2} + \frac{1}{\varepsilon^2} \nabla_y \cdot \nabla_y U^\varepsilon(x, y, t), & y \in Y_1, & x \in G, \\ (1.18.b) \quad \frac{\partial U^\varepsilon}{\partial t} &= \nabla_y \cdot \nabla_y U^\varepsilon(x, y, t) + \varepsilon^2 \frac{\partial^2 U^\varepsilon}{\partial x^2}, & y \in Y_2, & x \in G, \\ (1.18.c) \quad \nabla_y U^\varepsilon \cdot \nu|_{Y_1} &= \varepsilon^2 \nabla_y U^\varepsilon \cdot \nu|_{Y_2}, & y \in \Gamma_{11}, & x \in G, \\ (1.18.d) \quad \nabla_y U^\varepsilon \cdot \nu &= 0, & y \in \Gamma_{22}, & x \in G. \end{aligned}$$

Now we can recognize the limit as $\varepsilon \rightarrow 0$. In Y_1 we get $U(x, y, t) = u(x, t)$ in the limit, i.e., it is independent of $y \in Y_1$. Average (1.18.a) over Y_1 and use (1.18.c) with Gauss' theorem to get an integral over Γ_{11} . This gives the following limiting form of the problem:

$$\begin{aligned} (1.19.a) \quad \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} &= \frac{\partial^2 u}{\partial x^2} - \frac{1}{|Y_1|} \int_{\Gamma_{11}} \nabla_y U \cdot \nu \, ds, & x \in G, \\ (1.19.b) \quad \frac{\partial U}{\partial t} &= \nabla_y \cdot \nabla_y U(x, y, t), & y \in Y_2, & x \in G, \\ (1.19.c) \quad U(x, s, t) &= u(x, t), & s \in \Gamma_{11}, & x \in G, \\ (1.19.d) \quad \nabla_y U^\varepsilon \cdot \nu &= 0, & s \in \Gamma_{22}, & x \in G. \end{aligned}$$

The integral term in (1.19.a) is the total heat lost to the pipe wall at the cross-section $x \in G^\varepsilon$, computed from the normal component of the heat gradient in the wall, and it is comparable to the transport terms representing convection and conduction along the pipe. This balances the heat gained in the water with its moderate gradients against that exchanged with the pipe where the gradients are very high.

This problem approximates the heat transport and the exchange between the water and the pipe walls. It is essentially the classical *heat recuperator* problem, and it is the earliest example of a *distributed microstructure model*. The *global* domain

is the interval G . For each point $x \in G$ there is identified a *local cell*, Y_2 , which is located at or identified with that point. In the exchange of heat with the global medium, this family of cells acts as a distributed source, and the global medium is coupled to the cell at that point only on its boundary, Γ_{11} . It provides a well-posed problem which is a good approximation to the apparently singular ε -problem. We see the apparently singular term in (1.18.a) arose in the rescaling from the singular geometry in the original problem, but it is balanced by the corresponding term in (1.18.b) without the small coefficient because of the flux condition (1.18.c). In particular, the choice of ε^2 as coefficient in (1.18.b) exactly balanced the competing singularities introduced by the geometry and the materials.

A.2 Flow in Porous Media

A porous medium G in $\mathbb{R}^m$ is filled with a fluid, either liquid or gas, and this fluid diffuses from locations of higher to those of lower pressure. We begin by modeling this situation. Let $p(x, t)$ denote the *pressure* of fluid at the point $x \in G$ and time $t > 0$, and denote the corresponding *density* by $\rho(x, t)$. The quantity of fluid in an element of volume V is $\int_V c(x)\rho(x, t) dx$, and this defines the *porosity* $c(x)$ of the medium at the point x . This is the volume fraction of the medium that is accessible to the fluid. The *flux* is the vector flow rate $\vec{J}(x, t)$, so the rate at which fluid moves across a surface element S with normal $\vec{\nu}$ is given by $\int_S \vec{J}(x, t) \cdot \vec{\nu} dS$. Then the *conservation of fluid* takes the integral form

$$\frac{\partial}{\partial t} \int_{G_0} c(x)\rho(x, t) dx + \int_{\partial G_0} \vec{J} \cdot \vec{\nu} dS = \int_{G_0} f(x, t) dx, \quad G_0 \subset G,$$

in which $f(x, t)$ denotes any volume distributed *source density*. When the flux and density are differentiable, we can write this in the differential form

$$\frac{\partial}{\partial t} c(x)\rho(x, t) + \vec{\nabla} \cdot \vec{J}(x, t) = f(x, t), \quad x \in G.$$

The statement that the flux depends on the pressure gradient is *Darcy's law*, and it takes the form

$$\vec{J}(x, t) = -\frac{k(x)}{\mu} \rho \vec{\nabla} p(x, t).$$

This defines the *permeability* $k(x)$ of the porous medium. The value of k is a measure of the ease with which the fluid flows through the medium, and μ is the *viscosity* of the fluid. That is, $\frac{1}{k}$ is the *resistance* of the medium to flow, and μ is a corresponding property of the fluid. Finally, the type of fluid considered is described by the *equation of state*,

$$\rho = s(p).$$

The function $s(\cdot)$ which relates the pressure and density is monotone, in fact, it is usually chosen to be strictly increasing. However in problems with *partial saturation*, it is necessary to let $s(\cdot)$ be a *graph*, since at $p = 0$ fluid may only partially fill the pores and thereby give an effective density within the interval $[0, \rho(0)]$ related to the fraction of fluid present. By substituting the appropriate quantities above we obtain

$$\frac{\partial}{\partial t} c(x)s(p(x, t)) - \vec{\nabla} \cdot \frac{k(x)}{\mu} \vec{\nabla} S(p(x, t)) = f(x, t), \quad x \in G, \quad t > 0,$$

where we denote by $S(\cdot)$ the antiderivative of $s(\cdot)$. That is, we set $S(p) = \int_0^p s(r) dr$. If we introduce the variable $u = S(p)$ we obtain the *generalized porous medium equation*

$$(2.1) \quad \frac{\partial}{\partial t} c(x) a(u(x, t)) - \vec{\nabla} \cdot \frac{k(x)}{\mu} \vec{\nabla} u(x, t) = f(x, t), \quad x \in G, \quad t > 0,$$

with $a(\cdot) \equiv s(\cdot) \circ S^{-1}(\cdot)$.

The classical case is obtained by choosing an equation of state that is specific to the flow of gas. This corresponds to the choice of the function

$$s(p) = \rho_0 p^\alpha$$

in the equation of state where ρ_0 and α are positive constants with $\alpha \leq 1$. If the medium is homogeneous, so the porosity and permeability are independent of $x \in G$, then this leads to the *classical porous medium equation*

$$(2.2) \quad \frac{\partial}{\partial t} c\rho - \frac{k}{\mu(\alpha + 1)\rho_0^{m-1}} \Delta \rho^m = f,$$

where $m = 1 + \frac{1}{\alpha}$. Other situations in which $(\cdot)$ arises include boundary layer theory, where $m = 2$, population models, interstellar diffusion of galactic civilizations, and certain problems of plasma physics. The nonlinearity satisfies $m > 1$ in all but the last of these examples, and there $0 < m < 1$. The situation with $m > 1$ is called the *slow diffusion* case, and $0 < m < 1$ is the *fast diffusion* case. In the former case disturbances have a finite speed of propagation, unlike the linear case $m = 1$, and any solution which initially has compact support will continue to have compact support for all later times.

The simplest situation is that of a *slightly compressible* fluid. Here the equation of state has the form $s(p) = \exp c_0 p$ where $c_0 > 0$ is the *compressibility* of the fluid. Then (2.1) simplifies to the linear parabolic equation

$$(2.3) \quad \frac{\partial}{\partial t} c\rho - \frac{kc_0}{\mu} \Delta \rho = f.$$

This is formally the same as the heat equation, i.e., it is (2.2) with $m = 1$.

If we assume that the Darcy Law has the nonlinear form

$$\vec{J} = -\frac{k(\rho \vec{\nabla} p)}{\mu} \rho \vec{\nabla} p$$

in which the permeability depends on the flux, then in terms of the variable $u = S(p)$ we obtain the *quasilinear porous medium equation*

$$(2.4) \quad \frac{\partial}{\partial t} c(x) a(u(x, t)) - \vec{\nabla} \cdot \frac{k(\vec{\nabla} u(x, t))}{\mu} \vec{\nabla} u(x, t) = f(x, t), \quad x \in G, \quad t > 0,$$

with $a(\cdot) \equiv s(\cdot) \circ S^{-1}(\cdot)$ as before. Finally, if we specialize this to the case of a slightly compressible fluid, we obtain

$$(2.5) \quad \frac{\partial}{\partial t} c(x) c_0 u(x, t) - \vec{\nabla} \cdot \frac{k(\vec{\nabla} u(x, t))}{\mu} \vec{\nabla} u(x, t) = f(x, t), \quad x \in G, \quad t > 0.$$

Fissured Media. The flow of fluid through a fissured porous medium leads to some related systems. A fissured medium consists of a matrix of porous and permeable blocks of pores or *cells* which are isolated from each other by a highly developed system of *fissures* through which the bulk of the fluid transport occurs. Due to this separation of the cells by the fissure system, there is a negligible amount of transport directly from cell to cell. Another feature of fissured media is that the volume occupied by the fissures is considerably smaller than that occupied by the matrix of cells. Thus, most transport occurs in the fissures and the bulk of the storage takes place in the cells. The system is by nature very much unsymmetric in the structure and function of its components. Specifically, the fissure system provides the primary transport component and the cell system accounts for all storage. Thus, the exchange between the fissure system and the matrix of cells is an important component of the model.

The *double porosity model* is a classical description of diffusion in heterogeneous media. The idea is to introduce at each point in space a density, pressure or concentration for each component, each being obtained by averaging in the respective medium over a generic neighborhood sufficiently large to contain a representative sample of each component. The rate of exchange between the components must be expressed in terms of these quantities, and the resulting expressions become distributed source and sink terms for the diffusion equations in the individual components. Thus, one obtains a *system* of diffusion equations, one for each component. The classical linear double porosity model for the flow of slightly compressible fluid in a general heterogeneous medium consisting of two components is

$$(2.6) \quad \begin{aligned} \frac{\partial}{\partial t} c_1 u_1 - \frac{k_1 c_0}{\mu} \Delta u_1 + \frac{1}{\alpha} (u_1 - u_2) &= f, \\ \frac{\partial}{\partial t} c_2 u_2 - \frac{k_2 c_0}{\mu} \Delta u_2 + \frac{1}{\alpha} (u_2 - u_1) &= g. \end{aligned}$$

The first equation describes the flow in fissures — regions of small relative volume but large permeability. The second describes the flow in the cell system — regions of large porosity or volume, but largely isolated from one another. Both of these equations are to be understood macroscopically; that is, these equations are obtained by averaging over neighborhoods containing a large number of cells (called “blocks” of pores) and fissures. Although the components of (2.6) are structured symmetrically, fissured media characteristics are modeled by very small coefficients c_1 and k_2 .

For the fissured medium model, the coefficient c_1 is almost zero because the relative volume of the fissures is small, and $k_2 = 0$ because there is no direct flow from one block of pores to another, i.e., each cell is isolated from the adjacent cells by the fissure system. The last term on the left of each equation represents the exchange of fluid between the cells and the fissures. The parameter α represents the resistance of the medium to this exchange. When $\alpha = \infty$, no exchange flow is possible. An alternative interpretation is that $1/\alpha$ represents the degree of fissuring in the medium. When the degree of fissuring is infinite, the exchange flow encounters no resistance and $u_0 = u_1$. The external sources of fluid represented by f and g are located in the fissures and in the cells, respectively.

Consider the analogous quasilinear situation in which the permeability is flux-dependent. Then we obtain the system

$$(2.7) \quad \begin{aligned} \frac{\partial}{\partial t} c_1(x) a(u_1(x, t)) - \vec{\nabla} \cdot \frac{k_1(\vec{\nabla} u_1(x, t))}{\mu} \vec{\nabla} u_1(x, t) + \tau(u_1 - u_2) &= f(x, t), \\ \frac{\partial}{\partial t} c_2(x) a(u_2(x, t)) - \vec{\nabla} \cdot \frac{k(\vec{\nabla} u_2(x, t))}{\mu} \vec{\nabla} u_2(x, t) - \tau(u_1 - u_2) &= g(x, t), \end{aligned}$$

in terms of the variables $u_1 = S(p_1)$ and $u_2 = S(p_2)$. The fluid exchange rate between the cells and fissures is assumed to be determined by a monotone function $\tau(\cdot)$ of the pressure difference and the density, so it is given by $\tau(\bar{\rho}(p_1 - p_2))$. Here we choose $\bar{\rho}$ to be the *average* density on the pressure interval $[p_1, p_2]$, so we have

$$\bar{\rho}(p_2 - p_1) = \int_{p_1}^{p_2} s(r) dr = u_2 - u_1.$$

In this way we have obtained an exchange term which is a function of the difference of the components instead of a difference of functions of the two components. If we specialize this to the case of a slightly compressible fluid we obtain the system

$$(2.8) \quad \begin{aligned} \frac{\partial}{\partial t} c_1(x) c_0(u_1(x, t)) - \vec{\nabla} \cdot \frac{k_1(\vec{\nabla} u_1(x, t))}{\mu} \vec{\nabla} u_1(x, t) + \gamma(u_1 - u_2) &= f(x, t), \\ \frac{\partial}{\partial t} c_2(x) c_0(u_2(x, t)) - \vec{\nabla} \cdot \frac{k(\vec{\nabla} u_2(x, t))}{\mu} \vec{\nabla} u_2(x, t) - \gamma(u_1 - u_2) &= g(x, t). \end{aligned}$$

These provide examples of parabolic partial differential equations and systems with multiple nonlinearities as well as a variety of *types* of nonlinearity that can arise.

A.3 Continuum Mechanics

Our objective is to describe the formulation and the well-posedness of some classical problems of vibration of a body composed of visco-elastic or plastic material. Although these can easily be developed for a general body in R^m , we restrict our attention here for simplicity to the 1-dimensional case of longitudinal vibrations of a rod. The structure of the resulting system equations is the same, and these provide some interesting and instructive examples of evolution equations in Hilbert space.

Longitudinal vibrations. Consider the longitudinal vibrations in a long narrow cylindrical rod consisting of a material which is elastic and (possibly) viscous. We position the rod along the x -axis and identify it with the interval $[0, 1]$ in $\mathbb{R}$. The rod stretches or contracts in the horizontal direction, and we assume that the vertical plane cross-sections of the rod move only horizontally. Denote by $u(x, t)$ the *displacement* in the positive direction from the point $x \in [0, 1]$ at the time $t > 0$. The corresponding *displacement rate* or *velocity* is denoted by $v(x, t) \equiv u_t(x, t)$.

Let $\sigma(x, t)$ denote the local *stress*, the force per unit area with which the part of the rod to the left of the point x acts on the part to the right of x . For a section of the rod, $x_1 < x < x_2$, during the time interval, $t_1 < t < t_2$, the total impulse is given by

$$\int_{t_1}^{t_2} (\sigma(x_2, s) - \sigma(x_1, s)) ds.$$

If the density of the string at x is given by $\rho > 0$, the change in momentum of this section is just

$$\int_{x_1}^{x_2} \rho (u_t(x, t_2) - u_t(x, t_1)) dx .$$

If we let $F(x, t)$ denote any external applied force per unit of length in the positive x -direction, then we obtain from Newton's second law by equating the above that

$$\int_{x_1}^{x_2} \rho (u_t(x, t_2) - u_t(x, t_1)) dx = \int_{t_1}^{t_2} (\sigma(x_2, s) - \sigma(x_1, s)) ds + \int_{t_1}^{t_2} \int_{x_1}^{x_2} F(x, s) dx ds$$

for any such $x_1 < x_2$ and $t_1 < t_2$. For a sufficiently smooth displacement $u(x, t)$, the displacement therefore satisfies the *momentum equation*

$$(3.1.a) \quad \frac{\partial^2}{\partial t^2} (\rho u(x, t)) - \frac{\partial}{\partial x} \sigma(x, t) = F(x, t) , \quad 0 < x < 1 , \quad t > 0 .$$

The *stress* $\sigma(x, t)$ is determined by the type of material of which the rod is composed and the amount by which the neighboring region is stretched or compressed, i.e., on the *elongation* or *strain* $\varepsilon(x, t) \equiv \frac{\partial u(x, t)}{\partial x}$ at the point x . In the simplest case we assume that $\sigma(x, t)$ is proportional to $\varepsilon(x, t)$, i.e., that there is a constant E called *Young's modulus* for which

$$\sigma(x, t) = E \varepsilon(x, t) .$$

The constant E is a property of the material, and in this case we say the material is purely *elastic*. A rate-dependent component of the stress-strain relationship arises when the force generated by the elongation depends not only on the magnitude of the elongation but also on the speed at which it is changed, i.e., on the *elongation rate*

$$\frac{\partial \varepsilon(x, t)}{\partial t} = \frac{\partial v(x, t)}{\partial x} .$$

The simplest such case is that of a *visco-elastic* material defined by the linear *constitutive equation*

$$(3.1.b) \quad \sigma(x, t) = E \varepsilon(x, t) + \mu \frac{\partial \varepsilon(x, t)}{\partial t} ,$$

in which the material constant μ is the *viscosity* of the material.

The system (3.1) consisting of the momentum and constitutive equations together with appropriate boundary and initial conditions comprises a model for the simplest problems of elasticity theory, the one-dimensional case. Note that by substitution of (3.1.b) into (3.1.a), we obtain the *viscous wave equation*

$$(3.2) \quad \frac{\partial^2}{\partial t^2} (\rho u(x, t)) - \frac{\partial}{\partial x} \left(\mu \frac{\partial^2 u(x, t)}{\partial t \partial x} + E \frac{\partial u(x, t)}{\partial x} \right) = F(x, t) , \quad 0 < x < 1 , \quad t > 0 .$$

See Proposition I.6.1 and Corollary IV.6.3.

For the special case of $\mu = 0$ we first consider the system (3.1) in the form

$$(3.3.a) \quad v_t - E^{\frac{1}{2}} \sigma_x = F(x, t) ,$$

$$(3.3.b) \quad \sigma_t - E^{\frac{1}{2}} v_x = 0 .$$

(Here we have replaced σ by $E^{\frac{1}{2}} \sigma$ in order to obtain the symmetry.) We shall show that the dynamics is governed by a semigroup of contractions in a product of L^2

spaces. That is, the spatial part of this system is the realization of an *m-accretive* operator in the appropriate Hilbert space.

DEFINITION. The operator A is determined as follows: $[f, g] = A[v, \sigma]$ if

$$[f, g] \in L^2(0, 1) \times L^2(0, 1), \quad [v, \sigma] \in H^1(0, 1) \times H^1(0, 1),$$

and

$$\begin{aligned} -E^{\frac{1}{2}}\sigma_x(x) &= f(x), \quad 0 < x < 1, \quad \sigma(1) = 0 \\ -E^{\frac{1}{2}}v_x(x) &= g(x), \quad v(0) = 0. \end{aligned}$$

Here we have chosen for illustration boundary conditions associated with the requirements of no motion at the left end and no force at the right end of the rod. It is easy to check that A is accretive in $L^2(0, 1) \times L^2(0, 1)$. In fact, we have $(A[v, \sigma], [v, \sigma])_{L^2 \times L^2} = 0$. Define the Hilbert spaces

$$V = \{v \in H^1(0, 1) : v(0) = 0\}, \quad W = \{\sigma \in H^1(0, 1) : \sigma(1) = 0\}.$$

The corresponding resolvent equation, $(I + A)[v, \sigma] = [f, g]$, is given by

$$\begin{aligned} v \in V : \quad v - E^{\frac{1}{2}}\sigma_x &= f, \\ \sigma \in W : \quad \sigma - E^{\frac{1}{2}}v_x &= g. \end{aligned}$$

If we eliminate v , we can write this as a single equation or *variational equality*. To be precise, this problem has the following variational form: find a function

$$\sigma \in W : \quad \int_0^1 (\sigma w + E^{\frac{1}{2}}(E^{\frac{1}{2}}\sigma_x + f)w_x) dx = \int_0^1 g w dx, \quad w \in W.$$

In particular, this is the characterization of the solution to the problem of minimizing the convex function

$$\Phi(\sigma) = \int_0^1 \left(\frac{1}{2}\sigma^2 + \frac{E}{2}\sigma_x^2 \right) dx - \int_0^1 (g\sigma - f\sigma_x) dx$$

over the space W , so it is known to have a solution by Theorem I.2.1A and, hence, $\text{Rg}(I + A) = L^2(0, 1) \times L^2(0, 1)$. This shows that A is *m-accretive*, so Corollary I.5.2 and Corollary I.5.3 will show directly that there is a unique solution of (3.3) with

$$v \in C^1([0, T], V), \quad \sigma \in C^1([0, T], W).$$

The case with $\mu > 0$ is different. Here we have the corresponding system

$$(3.4.a) \quad v_t - E^{\frac{1}{2}}\sigma_x = F,$$

$$(3.4.b) \quad \sigma_t - E^{\frac{1}{2}}v_x - \frac{\mu}{E^{\frac{1}{2}}}\sigma_{xx} = \frac{\mu}{E^{\frac{1}{2}}}F_x,$$

and we cannot deduce from (3.4.b) that $v(t) \in H^1(0, 1)$. Note that if we had $E = 0$ we could substitute (3.1.b) directly into (3.1.a) to obtain the *heat equation*

$$\rho v_t - \mu v_{xx} = F(x, t)$$

for which we have not only well-posedness of the appropriate initial-boundary-value problem but also some *regularizing effects*.

In order to take advantage of the possibility to substitute for the viscous term, we write the system (3.1) in the form

$$\begin{aligned} \frac{\partial}{\partial t} v - \sigma_x &= F, \quad \sigma = E^{\frac{1}{2}} \sigma_1 + \mu^{\frac{1}{2}} \sigma_2, \\ \sigma_1 &= E^{\frac{1}{2}} \frac{\partial u}{\partial x}, \quad \sigma_2 = \mu^{\frac{1}{2}} \frac{\partial v}{\partial x}. \end{aligned}$$

This formulation displays the total stress as the sum of two components, the *parallel* addition of the elastic stress (corresponding to σ_1) with a purely viscous stress (corresponding to σ_2). By substituting the second back into the first equation and dropping the subscript on σ_1 , we obtain the system

$$(3.5.a) \quad v_t - E^{\frac{1}{2}} \sigma_x - \mu v_{xx} = F(x, t),$$

$$(3.5.b) \quad \sigma_t - E^{\frac{1}{2}} v_x = 0.$$

Note that we have moved the viscosity term from the second line of (3.4) to the first line of the system (3.5).

We shall show that the corresponding operator A is m -accretive in the space $H \equiv L^2(0, 1) \times L^2(0, 1)$. To this end, as well as to motivate our notation in the next section, we introduce the (unbounded) operator on $L^2(0, 1)$

$$D = \frac{d}{dx} : V \rightarrow L^2(0, 1)$$

and its *adjoint* operator

$$D^* = -\frac{d}{dx} : W \rightarrow L^2(0, 1).$$

DEFINITION. The operator A on $L^2(0, 1) \times L^2(0, 1)$ is determined as follows: $[f, g] \in A[v, \sigma]$ if

$$[f, g] \in L^2(0, 1) \times L^2(0, 1), \quad [v, \sigma] \in V \times W,$$

and

$$\begin{aligned} D^*(E^{\frac{1}{2}} \sigma + \mu Dv) &= f, \\ -E^{\frac{1}{2}} Dv &= g. \end{aligned}$$

Note that it is implicit in the definition that the total stress satisfies $E^{\frac{1}{2}} \sigma + \mu Dv \in W$. The corresponding resolvent equation, $(I + A)[v, \sigma] = [f, g]$, is given by

$$\begin{aligned} v \in V : \quad v + D^*(E^{\frac{1}{2}} \sigma + \mu Dv) &= f, \\ \sigma \in W : \quad \sigma - E^{\frac{1}{2}} Dv &= g. \end{aligned}$$

We compute

$$(A[v, \sigma], [v, \sigma])_{L^2 \times L^2} = \mu(Dv, Dv)_{L^2},$$

so A is accretive. The resolvent equation is equivalent to the single equation

$$v + D^*(EDv + E^{\frac{1}{2}} g + \mu Dv) = f,$$

that is, to the variational equation

$$v \in V : \quad (v, \varphi)_{L^2 \times L^2} + ((E + \mu)Dv + E^{\frac{1}{2}} g, D\varphi)_{L^2 \times L^2} = (f, \varphi)_{L^2 \times L^2}, \quad \varphi \in V,$$

and this is guaranteed by Theorem I.2.1A to have a unique solution if $E + \mu > 0$. Thus the dynamic problem has a unique solution by Corollary I.5.2.

Nonlinear Models.

We consider briefly the modifications above in the case that the constitutive equation (3.1.b) is nonlinear. First we note that if the elasticity term were nonlinear, then we obtain a *quasilinear wave equation*

$$\frac{\partial^2}{\partial t^2}(\rho u(x, t)) - \frac{\partial}{\partial x}\left(E\left(\frac{\partial u(x, t)}{\partial x}\right)\right) = F(x, t).$$

Such equations are not handled by monotone methods, and we have not discussed them. However in the case of a *purely viscous* material, we obtain the *quasilinear diffusion equation*

$$\frac{\partial \rho v}{\partial t} - \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial v}{\partial x}\right)\right) = F(x, t)$$

with the monotone viscosity function, $\mu(\cdot)$. Such problems were resolved by Proposition IV.5.2 in Example IV.5.A. More generally, if we substitute (3.1.b) with such a viscosity term into (3.1.a), we obtain the *quasilinear-damped wave equation*

$$\frac{\partial^2}{\partial t^2}(\rho u(x, t)) - \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial^2 u(x, t)}{\partial x \partial t}\right) + E\frac{\partial u(x, t)}{\partial x}\right) = F(x, t),$$

or equivalently

$$\frac{\partial^2}{\partial t^2}(\rho v(x, t)) - \frac{\partial}{\partial t}\frac{\partial}{\partial x}\left(\mu\left(\frac{\partial v(x, t)}{\partial x}\right) + E\frac{\partial v(x, t)}{\partial x}\right) = \frac{\partial}{\partial t}F(x, t), \quad 0 < x < 1, \quad t > 0.$$

See Theorem IV.6.2 and the abstract equations IV.6.13 and IV.6.14.

Plasticity Models.

We describe two models of plasticity in very simple form. For each of these examples, we shall describe the operator in L^2 that realizes the corresponding initial-boundary value problem. The general approach is to express the constitutive relation (3.1.1.b) as a variational equation of evolution type

$$(3.6) \quad \sigma_t + \partial\varphi(\sigma) \ni Dv$$

whose input, the strain-rate $\frac{\partial \varepsilon}{\partial t} = Dv$ corresponding to the displacement-rate $v = u_t$ is in L^2 . Here $\varphi(\cdot)$ denotes either the indicator function $I_K(\cdot)$ of a given closed convex set K characterizing the particular plasticity model or a smooth convex function for the viscosity models, and $\partial\varphi$ is the corresponding subgradient or derivative, respectively. The existence and uniqueness of a solution $\sigma(t)$, $t > 0$ of (3.6) with prescribed initial value $\sigma(0)$ is guaranteed by Theorem IV.4.3.

Elastic-perfectly Plastic. Consider a material whose stress-strain relationship is described as follows. When we apply an increasing load to one end of a segment of the rod material, there is initially no elongation within the segment, and the resulting stress at the other end, σ , similarly increases until the value $\sigma = 1$ is reached. Then the material *yields*, and any additional increase in the elongation ε occurs with no additional stress at the other end. If the load is incrementally decreased from $\sigma = 1$, the elongation remains fixed and the stress decreases until the value -1 is reached. Then the material again yields with decreasing elongation

ε and limited stress at the value $\sigma = -1$. This is a *perfectly plastic* material, and its constitutive relation is given by

$$\sigma \in \operatorname{sgn}\left(\frac{d\varepsilon}{dt}\right),$$

where the *sign graph* is given by

$$\operatorname{sgn}(x) = \begin{cases} \{1\}, & \text{if } x > 0, \\ [-1, 1], & \text{if } x = 0, \\ \{-1\}, & \text{if } x < 0. \end{cases}$$

If we add the perfectly elastic to the perfectly plastic model in *series*, then their elongations add, and we obtain the material defined by the relation

$$\sigma_t + \operatorname{sgn}^{-1}(\sigma) \ni E \varepsilon_t.$$

This is the *Prandtl* model of elasto-plasticity. Hereafter we take $E = 1$. The phase diagram showing the relationship between stress σ and strain ε is given in Figure 1. Since the value of σ depends on the past history of ε and is rate-independent, this relationship is a *hysteresis* functional.

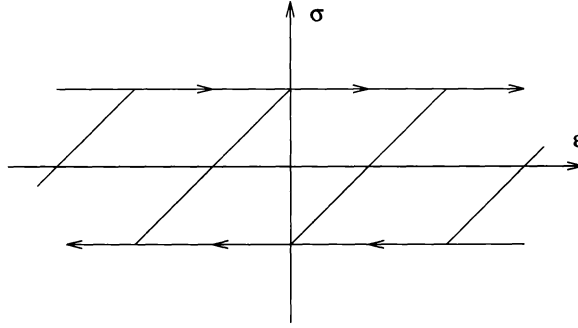


FIGURE 1

The resulting dynamical system is given by

$$(3.7.a) \quad v_t - \sigma_x = f, \quad 0 < x < 1, \quad 0 < t, \quad v(0, t) = 0$$

$$(3.7.b) \quad \sigma_t - v_x + \operatorname{sgn}^{-1}(\sigma) \ni 0, \quad \sigma(1, t) = 0$$

with appropriate initial conditions on v and σ . From the unbounded operator $D^* = -\frac{d}{dx} : W \rightarrow L^2(0, 1)$ we construct its *continuous dual*

$$(D^*)' : L^2(0, 1) \rightarrow W'.$$

Denote by $I_1(\cdot)$ the *indicator function* of the interval $[-1, 1]$. Thus, $I_1 : \mathbb{R} \rightarrow +\mathbb{R}_\infty$ is convex, proper, and lower-semi-continuous, and its subgradient is characterized as follows: $f \in \partial I_1(x)$ means

$$|x| \leq 1 \text{ and } \begin{cases} f \geq 0, & \text{for } x = 1, \\ f = 0, & \text{for } -1 < x < 1, \\ f \leq 0, & \text{for } x = -1. \end{cases}$$

Thus, ∂I_1 is just the inverse of the *sign graph*. Let the function φ_1 be the corresponding realization on the Hilbert space W (see Example II.8.B) given by

$$\varphi_1(w) = \int_0^1 I_1(w(x)) dx, \quad w \in W.$$

DEFINITION. The (weak) operator A is determined as follows: $[f, g] \in A[v, \sigma]$ if $[f, g] \in L^2(0, 1) \times L^2(0, 1)$, $[v, \sigma] \in L^2(0, 1) \times W$, and there exists a $c \in W'$ for which

$$(3.8.a) \quad \sigma \in W, \quad D^* \sigma = f,$$

$$(3.8.b) \quad v \in L^2(0, 1), \quad -(D^*)'v + c = g, \quad c \in \partial\varphi_1(\sigma).$$

REMARKS. The first term in (3.8.b) is defined in W' by

$$-v(D^*w) = \int_0^1 v(x)w'(x) dx, \quad w \in W.$$

Formally (3.8.b) means

$$-v_x(x) + c(x) = g(x), \quad c(x) \in \operatorname{sgn}^{-1}(\sigma(x)), \quad v(0) = 0,$$

but this holds only if c is sufficiently regular, e.g., if $c \in L^2(0, 1)$, for then $v \in H^1(0, 1)$ and the boundary condition is meaningful. The range, $\operatorname{Rg}(A)$, is the set of pairs $[f, g] \in L^2(0, 1) \times L^2(0, 1)$ for which $|\int_x^1 f dx| \leq 1$ for each $x \in [0, 1]$. Neither A nor A^{-1} is a function.

The corresponding resolvent equation, $(I + A)[v, \sigma] \ni [f, g]$, is given by

$$\begin{aligned} v \in L^2(0, 1) : \quad v + D^* \sigma &= f, \\ \sigma \in W : \quad \sigma - (D^*)'v + \partial\varphi_1(\sigma) &\ni g. \end{aligned}$$

Note that $(D^*)'v$ and $\partial\varphi_1(\sigma)$ are in W' , so this is a *weak* solution in our notation below, and there is no boundary value assigned to $v(0)$. If we eliminate v , we can write this as a single equation or *variational inequality*, formally of the form

$$\sigma \in W : \quad \sigma + \partial\varphi_1(\sigma) + (D^*)'(D^* \sigma) \ni g + (D^*)'f.$$

To be precise, this problem has the following variational form: find a pair of functions

$$\begin{aligned} \sigma \in W, \quad c \in W' \text{ such that } c \in \partial\varphi_1(\sigma), \text{ and} \\ \int_0^1 (\sigma\varphi + (\sigma_x + f)\varphi_x) dx + c(\varphi) = \int_0^1 g\varphi dx \text{ for all } \varphi \in W. \end{aligned}$$

This is the characterization of the solution to the problem of minimizing the Dirichlet integrand

$$\Phi(\sigma) = \int_0^1 \left(\frac{1}{2} \sigma^2 + \frac{1}{2} \sigma_x^2 + I_1(\sigma) \right) dx - \int_0^1 (g\sigma - f\sigma_x) dx$$

over the space W . This is contained in Example II.8.D (also see Proposition IV.1.4 or Proposition IV.1.5), so it is known to have a solution and, hence, $\operatorname{Rg}(I + A) = L^2(0, 1) \times L^2(0, 1)$. It follows as before that the map $[f, g] \rightarrow [v, \sigma]$ is a contraction

on $L^2(0, 1) \times L^2(0, 1)$, so A is m-accretive, and Theorem IV.4.1 shows that there is a unique *weak* solution of (3.7) with

$$\frac{\partial v}{\partial t}, \frac{\partial \sigma}{\partial t} \in L^\infty(0, T; L^2(0, 1)) , \quad v \in L^\infty(0, T; V), \sigma \in L^\infty(0, T; W) .$$

REMARK. The corresponding equation for v is degenerate in the gradient, hence, not coercive.

Kinematic Hardening. Here we assume that the material *work-hardens* each time the yield stress is reached. In this case the *position* of the stress interval is moved upward or downward. Momentum and constitutive equations are, respectively,

$$(3.9) \quad \begin{aligned} \frac{\partial}{\partial t} v - \sigma_x &= f , \quad \sigma = \dot{\beta}_1 \sigma_1 + \beta_2 \sigma_2 , \\ \frac{\partial}{\partial t} \sigma_1 + \partial \varphi_1(\sigma_1) &\ni \beta_1 \frac{\partial}{\partial x} v , \quad \frac{\partial}{\partial t} \sigma_2 = \beta_2 \frac{\partial}{\partial x} v . \end{aligned}$$

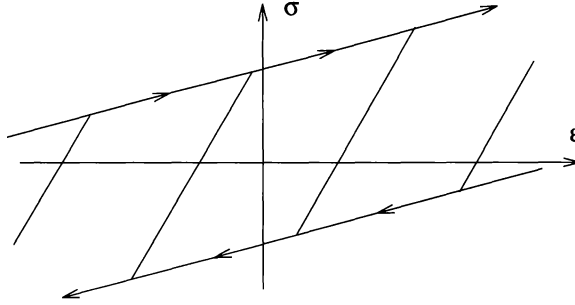


FIGURE 2

This model results from the *parallel* addition of the elastic-plastic stress from above (corresponding to σ_1) with a purely elastic stress (corresponding to σ_2) which records the position of the center of the yield stress interval. Thus the lines in Figure 2 representing the upper and lower yield surfaces are at a vertical distance apart of 2, and they have slope β_2 .

We shall write the system (3.9) as an evolution equation in the appropriate product space.

DEFINITION. The (strong) operator A is determined as follows:
 $[f, g_1, g_2] \in A[v, \sigma_1, \sigma_2]$ if

$$[f, g_1, g_2] \in L^2(0, 1)^3, \quad [v, \sigma_1, \sigma_2] \in V \times L^2(0, 1)^2, \quad \beta_1 \sigma_1 + \beta_2 \sigma_2 \in W ,$$

and there exists a $c \in L^2(0, 1)$ for which

$$(3.10) \quad \begin{aligned} D^*(\beta_1 \sigma_1 + \beta_2 \sigma_2) &= f , \\ -\beta_1 Dv + c &= g_1 , \quad c \in \partial \varphi_1(\sigma_1) , \\ -\beta_2 Dv &= g_2 . \end{aligned}$$

Note that since $c \in L^2(0, 1)$, the inclusion $c(x) \in \partial\varphi_1(\sigma_1(x))$ is a pointwise variational inequality in $\mathbb{R}$ for a.e. $x \in [0, 1]$. Namely, it is equivalent to

$$c(x) \in \mathbb{R}, \quad |\sigma_1(x)| \leq 1 : \quad c(x)(\rho - \sigma_1(x)) \leq 0 \text{ for all } \rho \in \mathbb{R} \text{ with } |\rho| \leq 1,$$

the sign graph. We shall show that the operator A is m -accretive in the space $H \equiv L^2(0, 1)^3$ and, since $v_x \in L^2(0, 1)$, that it leads to a *strong* solution.

We introduce the following:

$$\beta = [\beta_1 I, \beta_2 I] : L^2(0, 1) \rightarrow L^2(0, 1)^2 \text{ where } \beta_1, \beta_2 \in \mathbb{R} \text{ are given,}$$

$$\beta^*[\sigma] = \beta_1 \sigma_1 + \beta_2 \sigma_2, \quad \beta^* : L^2(0, 1)^2 \rightarrow L^2(0, 1)$$

$$W_0 = \{\sigma = [\sigma_1, \sigma_2] \in L^2(0, 1)^2 : \beta^* \sigma \in H^1(0, 1), \beta^* \sigma(1) = 0\}$$

Note that for any solution of (3.9), either weak or strong, we have $\beta^* \sigma \in W$ in the momentum equation. In particular, $\beta^* \sigma$ satisfies the appropriate boundary condition.

First we check by a direct estimate that A is accretive. Second, the resolvent equation $(I + A)[v, \sigma] \ni [f, g]$ is equivalent to solving the system

$$v \in V : \quad v + D^* \beta^* \sigma = f,$$

$$\sigma \in W_0 : \quad \sigma - \beta Dv + [\partial\varphi_1(\sigma_1), 0] = g \in L^2(0, 1)^2.$$

This is equivalent to solving for v the equation

$$v \in V : \quad v + D^*(\beta_1(I + \partial\varphi_1)^{-1}(\beta_1 Dv + g_1) + \beta_2^2 Dv + \beta_2 g_2) = f \text{ in } V'.$$

Since $\beta_2^2 > 0$, the form is coercive, and existence of a solution follows. (See Example II.8.D.) The components of $[\sigma_1, \sigma_2] \in W_0$ are obtained directly from the second and third terms in this equation, respectively, and then we check that $\sigma \in W_0$. In particular, the boundary condition at $x = 1$ is satisfied. These remarks show that Theorem IV.4.1 applies directly to give existence and uniqueness of a *strong* solution of (3.9) with

$$v \in L^\infty(0, T; V), \quad \sigma \in L^\infty(0, T; W_0), \quad \frac{\partial v}{\partial t}, \frac{\partial \sigma}{\partial t} \in L^\infty(0, T; L^2(0, 1)).$$

REMARK 1. Since v belongs to V instead of merely to $L^2(I)$, the solution here is *smoother* than that above. This is made possible here by the coercivity resulting from the β_2 term.

REMARK 2. $A(v, \sigma)$ is single valued only if $\sigma_1 \neq 0$ a.e., and $A^{-1}(f, g)$ is single valued only if $\beta_2 g_1 \neq \beta_1 g_2$ a.e..

REMARK 3. We can include a viscous element in parallel to the above by adding a third equation of the form

$$\frac{1}{k} \frac{\partial}{\partial t} \sigma_3 + \frac{1}{\mu} \sigma_3 = \beta_3 \frac{\partial}{\partial x} v.$$

More generally, we can include *visco-elastic* elements in the form

$$\frac{1}{k} \frac{\partial}{\partial t} \sigma_3 + J'(\sigma_3) = c \frac{\partial}{\partial x} v.$$

where J has a *bounded* derivative. This represents a series combination of elastic element and a purely viscous element, and one obtains *strong* solutions as above.

REMARK 4. By setting $v = u_t$, eliminating $\sigma_2 = \beta_2 \frac{\partial}{\partial x} u$, and replacing the term $\partial\varphi_1(\sigma_1)$ by $-\Delta\sigma_1$ in (3.9), we obtain the system

$$\begin{aligned} u_{tt} - \beta_1 \frac{\partial}{\partial x} \sigma_1 - \beta_2^2 u_{xx} &= f, \\ \frac{\partial}{\partial t} \sigma_1 - \Delta\sigma_1 &= \beta_1 \frac{\partial}{\partial x} u_t. \end{aligned}$$

This is the classical problem of *thermoelasticity*, and it is easier and can be handled similarly.

We close with a simple but important extension of the preceding example to a plasticity model built on four stress components. This will motivate the consideration of generalized sums or *integrals* of a collection or even a *continuum* of such components. The system is given by

$$\begin{aligned} \frac{\partial}{\partial t} v - \sigma_x &= f, \quad \sigma = \sigma_1 + \frac{1}{2} \sigma_2 + \frac{1}{4} \sigma_3 + \frac{1}{4} \sigma_4, \\ \frac{\partial}{\partial t} \sigma_1 + \partial\varphi_1(\sigma_1) &\ni \frac{\partial}{\partial x} v, \\ \frac{\partial}{\partial t} \sigma_2 + \partial\varphi_2(\sigma_2) &\ni \frac{1}{2} \frac{\partial}{\partial x} v, \\ \frac{\partial}{\partial t} \sigma_3 + \partial\varphi_3(\sigma_3) &\ni \frac{1}{4} \frac{\partial}{\partial x} v, \\ \frac{\partial}{\partial t} \sigma_4 &= \frac{1}{4} \frac{\partial}{\partial x} v. \end{aligned} \tag{3.11}$$

For each $j = 1, 2, 3$, φ_j is the indicator function of the interval $[-j, j]$, so the corresponding stress component σ_j is constrained to lie within that interval. The relation between total stress σ and strain ε is indicated by Figure 3. For an example of a typical output of this system, we begin with all components at 0, increase the strain, ε , from 0 to 5, decrease it to -5 , then increase it to 2, and we follow the resulting stress, σ . The slope of the stress starting upward from the origin is 2, then it decreases to 1 and to $\frac{1}{2}$ on successive intervals until only σ_4 with slope $\frac{1}{4}$ is active for $\varepsilon \geq 3$. When the curve begins to decrease from $\varepsilon = 5$, the slope is initially 2, and then the slope decreases successively to 1 and to $\frac{1}{2}$ on intervals of length 2 until only σ_4 with slope $\frac{1}{4}$ is active for $\varepsilon \leq -1$. The applied strain ε reverses direction again at -5 , and the resulting stress begins to rise with slope 2 again. The limiting positive slope $\frac{1}{4}$ is the *work-hardening* component, and it is this component of the stress that will lead to a *strong* solution of (3.11) as before.

Since the bounding lines in this hysteresis functional are straight lines, such models are called *multilinear*. By using a collection of such components, one can approximate a large class of convex bounding curves; with a continuum of such components, the corresponding class of convex functions can be matched. Most models of plasticity involve such multiple yield surfaces, and these provide an approximation of the observed smooth transitions between elastic and plastic regimes. Such smooth transitions are best modeled by a continuum of elastic-plastic elements with varying yield surfaces.

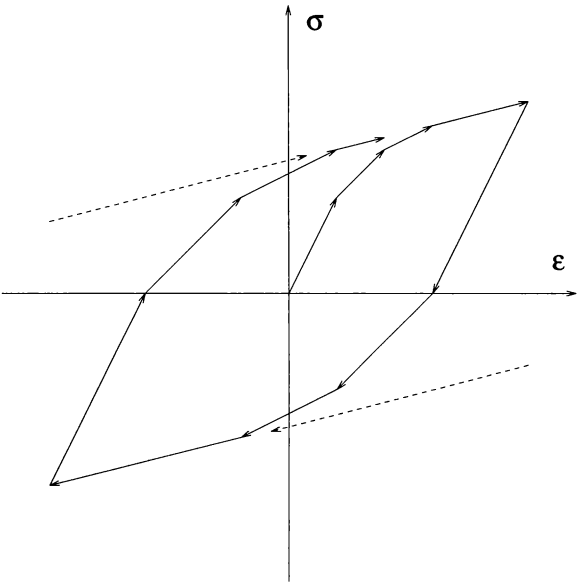


FIGURE 3

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